Week 9

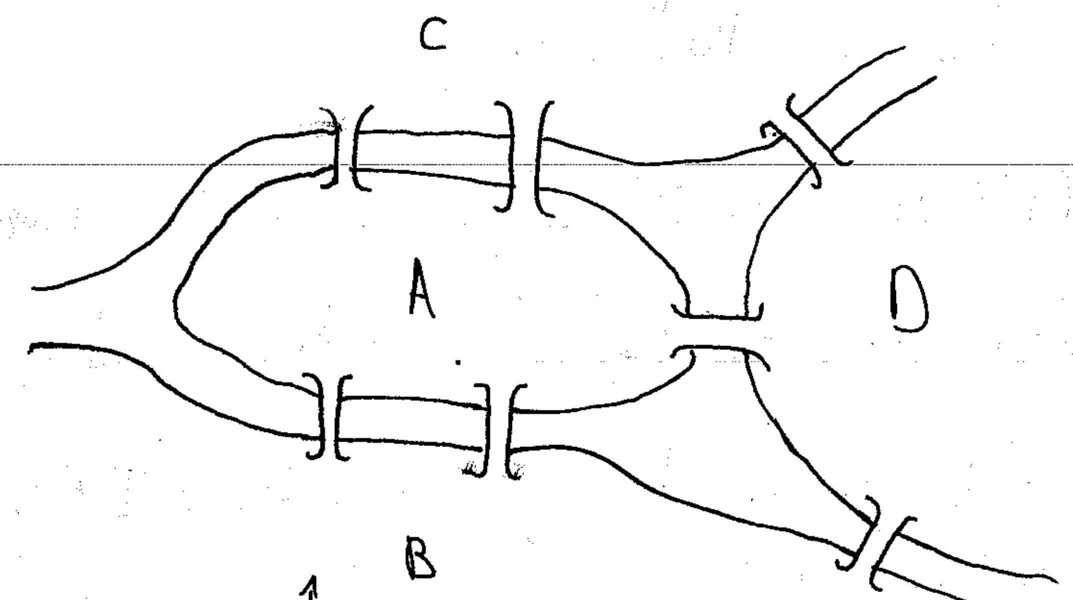
Discrete Mathematics 1

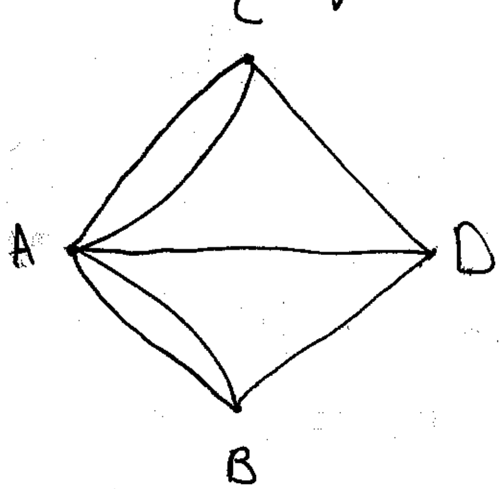
Incidence Structures

0. Motivation

To abstract symmetric and non-symmetric relations and study their combinatorial structures and properties:

In 1735, Euler solved the "Bridges of Königsberg Problem":





"Find a path going through each bridge exactly once that comes back to its starting point."

⇝ The precise map is not relevant but only the 'adjacency" or "incidence" relations between pieces of land & se.

1. Basic notions

Let $V := [n]$ be a set of nodes, or vertices, or points, with $n \geq 1$.

Def: A (general) incidence structure on V is a map $I : 2^{[n]} \longrightarrow \mathbb{N}$ such that $I(\emptyset) = 1$

This definition is very general, but allows to make precise many structures and their relations.

If $I(2^{[n]}) \subseteq \{0, 1\}$, I is a (usual) incidence structure.

↳ In this case $L := I^{-1}(1) \setminus \{\emptyset\}$ are called lines.

Equivalently, I can be defined as a relation $I \subseteq V \times L$ between vertices and lines.

Def: The incidence matrix M_I of a usual incidence structure is a $(n \times q)$-matrix whose entries are given as follows:

$$M_I := \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccc} l_1 & l_2 & \cdots & l_q \end{array} \\ \begin{pmatrix} & & & \\ & & & \\ & m_{i,l_j} & & \\ & & & \end{pmatrix} \end{array} \qquad L = \{l_1, \ldots, l_q\} \subseteq 2^{[n]}$$

$$m_{i,l_j} = \begin{cases} 0 & \text{if } i \notin l_j \\ 1 & \text{else} \end{cases}$$

For general case, when $I(l_i) \geq 2$, the column l_i would be repeated "l_i" times.

Def: An <u>abstract simplicial complex</u> is an incidence structure $\Delta : 2^{[n]} \to \{0,1\}$ such that if $\Delta(f)=1$ and $g \subseteq f$, then $\Delta(g) = 1$.

In other words, Δ is a lower ideal of $(2^{[n]}, \subseteq)$.

Def: An incidence structure is q-<u>uniform</u> when there exists $q \geq 1$ such that $q = |l|$, $\forall\, l \in L$.

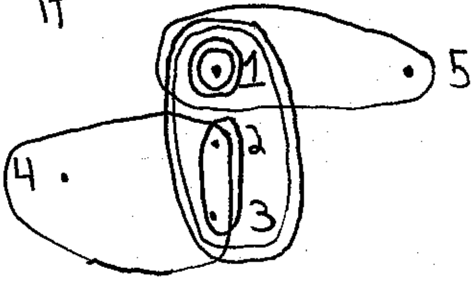
Another name for general incidence structure is <u>Hypergraph</u>.

<u>Examples</u>

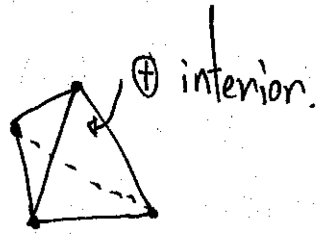
1) Let $n = 5$ and $M_I = \begin{pmatrix} 1&1&1&1&0&1&0 \\ 0&0&1&1&1&0&1 \\ 0&0&1&1&1&0&1 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \end{pmatrix}$

- Not uniform
- Contains multiple times the same subset.

We can draw it



2) Let $n = 4$ and $M_I = \begin{pmatrix} 1&0&0&0 & & 1 \\ 0&1&0&0 & \cdots \text{all subset} & 1 \\ 0&0&1&0 & & 1 \\ 0&0&0&1 & & 1 \end{pmatrix} \Rightarrow$



3) Let $n = 4$ and $M_I = \begin{pmatrix} 1&1&1&0&0&0 \\ 1&0&0&1&1&0 \\ 0&1&0&1&0&1 \\ 0&0&1&0&1&1 \end{pmatrix} \Rightarrow$

- 2-Uniform
- no multiple subset.

<u>Def1</u>: • A 2-uniform hypergraph is called a <u>graph</u>.

• A graph $G: 2^{[n]} \to \mathbb{N}$ is <u>simple</u> if $G(2^{[n]}) = \{0,1\}$, else it is called a <u>multigraph</u>.

•

Equivalent definition:

<u>Def2</u>: (Graph) Let $n \geq 1$ and $V := [n]$ and $E \subseteq \binom{[n]}{2}$.

The pair $G = (V, E)$ is called a <u>simple graph</u>.

If E is a multiset then G is a <u>multigraph</u>.

What about loops?

<u>Def 3</u>: (Graph allowing loops) Let $n \geq 1$ and $V := [n]$.

A graph G is given by a symmetric function $f: V \times V \to \mathbb{N}$.

The set $E := f^{-1}(\mathbb{N} \setminus \{0\})$ are the <u>edges</u> of G.

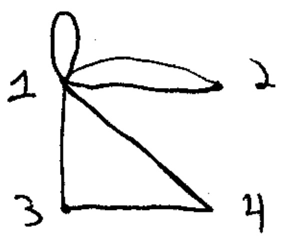
An element $(v,v) \in E$ is called a <u>loop</u>.

A graph G where $f(V \times V) \subseteq \{0,1\}$ and E does not contain loops is a <u>simple graph</u>.

All three definitions of simple graphs are equivalent.

Examples:

1) $V = [4]$



$f : [4] \times [4] \to \mathbb{N}$

$(1,2) = 2$ ← parallel or multi-edge.

$(1,1) = 1$ ← loop.

$(1,3) = 1$

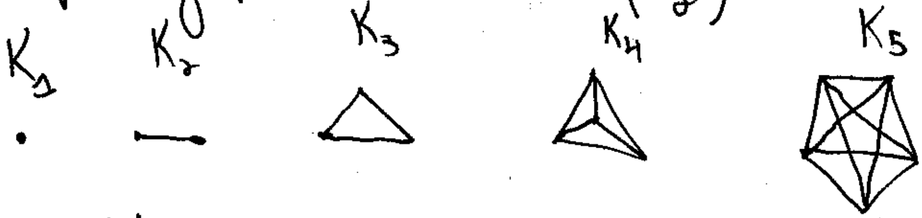
$(1,4) = 1$

$(3,4) = 1$

2) Empty graph: When $|E| = 0$.

This graph only has vertices but no edges.

3) Complete graph: When $E = \binom{[n]}{2}$.

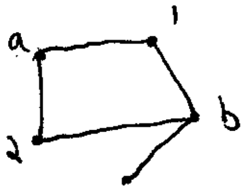


etc.

4) Bipartite graph: $|U_1| \neq 0 \neq |U_2|$

Let $V = U_1 \sqcup U_2$ and $G = (V, E)$ be such that $E \subseteq U_1 \times U_2$.

G is called bipartite.



$V = \{a, b\} \sqcup \{1, 2, 3\}$.

A complete bipartite is when $E = U_1 \times U_2$.

Similarly, one can define k-partite graphs when $V = U_1 \sqcup U_2 \sqcup \dots \sqcup U_k$ and $E \subseteq \bigcup_{i \neq j} U_i \times U_j$.

Convention: Unless otherwise stated a graph will always be simple.

Def: (Adjacence, Incidence)

Let $G=(V,E)$ be a graph and $e=(v_1,v_2)\in E$.

We say that
- v_1 is adjacent to v_2 (and vice-versa)
- v_1 and v_2 are incident to e (and vice-versa from context) (not always symmetric)

Def: (Order, Size)

- The order of a graph is $n=|V|$.
- The size of a graph is $q=|E|$.

Def: (Adjacency matrix)

The Adjacency matrix A_G of a graph $G=(V,E)$ is

$$A_G := \left(a_{u,v}\right)_{u,v\in V} \quad \text{where } a_{u,v} = \begin{cases} 0 & \text{if } (u,v)\notin E \\ 1 & \text{if } (u,v)\in E. \end{cases}$$

One can extend this definition to incidence structures. (no loops)

$$A_I := \left(a_{i,j}\right)_{(i,j)\in[n]^2} \quad \text{where } a_{i,j} = \#\text{ lines containing } i \text{ and } j.$$

<u>Examples:</u>

1) $M_I = \begin{pmatrix} 2 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ $\quad A_I = \begin{pmatrix} 5 & 2 & 2 & 0 & 1 \\ 2 & 4 & 4 & 1 & 0 \\ 2 & 4 & 4 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} = M_I \cdot M_I^T - \begin{pmatrix} \deg(1) & & & \\ & \deg(2) & & \\ & & \ddots & \\ & & & \deg(5) \end{pmatrix}$

<u>Def:</u> The <u>degree</u> $\deg(v)$ of a vertex $v \in V$ is the number of edges containing v (when the graph has loops it is counted twice)

2) <u>Empty graph:</u> $M_{\mathbb{E}} = \emptyset$ or $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ $A_{\mathbb{E}} = (0)_{n \times n}$

3) <u>Complete graph:</u> $M_{K_n} = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}$

4) <u>Complete Bipartite Graph:</u>

$$M_{K_{n \times n}} \quad \begin{array}{c|c} & \begin{array}{cc} u_1 & \quad u_2 \end{array} \\ \begin{array}{c} u_1 \\ \\ u_2 \end{array} & \left(\begin{array}{c|c} 0 & \begin{matrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{matrix} \\ \hline \begin{matrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{matrix} & 0 \end{array}\right) \end{array}$$

For simple graph, the diagonal of M_G is zero and symmetric.

<u>Def:</u> A graph is r-<u>regular</u> if $a \leq r = \deg(v)$, $\forall\, v \in V$.

Lemma: (Handshake Lemma)

Let $G=(V,E)$ be a graph. Then

$$\sum_{u\in V} \deg(u) = 2|E|.$$

In general, let $H=(V,E)$ be a q-uniform hypergraph, then

$$\sum_{u\in V} \deg(u) = q\cdot |E|.$$

Proof: (Double-counting)

Let $S=\{(v,e)\in V\times E \mid v\in e\}$

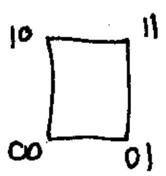
First, since a graph is 2-uniform, every edge provides two elements in S. Hence $|S|=2\cdot|E|$.

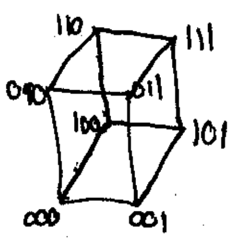
On the other hand, every vertex u in V provides $\deg(u)$ pairs in S.

The proof for hypergraphs is similar. ⊠

Corollary: Every graph has an even number of vertices of odd degree.

Example: Let $V=\{0,1\}^n$. Then $|V|=2^n$. Let $E\subseteq V\times V$ consists of the pairs (v_1,v_2) where v_1 and v_2 differ by exactly 1 digit.





$Q_n=(V,E)$: the hypercube graph.

The graph Q_n is $(n-1)$-regular. Hence by the lemma:

$$(n-1)\cdot 2^n = 2|E| \Leftrightarrow |E| = (n-1)\,2^{n-1}.$$

Looks familiar?

<u>Def:</u> (Hypergraph Duality)

Let $H=(V,E)$ be an hypergraph and M_H its incidence matrix. The <u>hypergraph dual</u> H^* of H has the transpose M_H^t as its incidence matrix.

By hypergraph duality we get the following dictionary

H hyper-graph	$\leftarrow$ Duality $\rightarrow$	H^* hyper graph
q-uniform	$\longleftrightarrow$	q-regular
r-regular	$\longleftrightarrow$	r-uniform.

<u>Lemma:</u> (Hypergraphs are bipartite graphs)

Let $H: 2^{[n]} \to \mathbb{N}$ be an hypergraph.

Let $V := [n] \sqcup \{\ell : \text{line of } H \text{ <u>with</u> multiplicity}\}$

and $E \subseteq [n] \times \{\ell : \ldots\}$ consist of the (i,ℓ) whenever $i \in \ell$ in H.

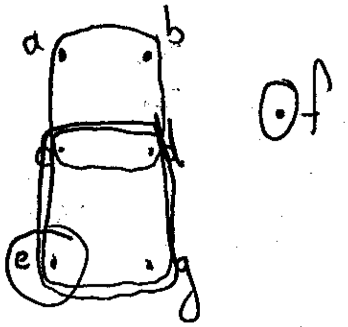
Then $G=(V,E)$ is a bipartite graph.

Example

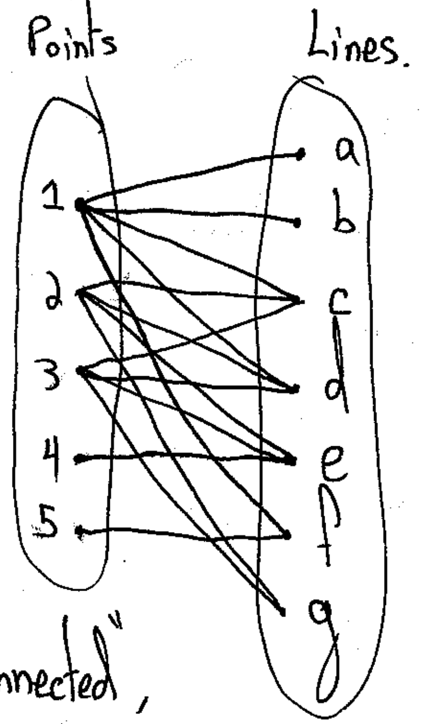
$$M_H = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array}$$
$$M_H = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

H^* its dual

$$M_{H^*} = \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$



The assaciated bipartite graph



If the bipartite graph is "connected", we can reverse the operation.

Def: A chain $(e_1, e_2, \dots, e_\ell)$ is a sequence of edges of $G=(V,E)$ such that $e_1=(v_0,v_1)$, $e_2=(v_1,v_2)\dots$, $e_\ell=(v_{\ell-1},v_\ell)$ for $v_i\in V$.

A chain is
- edge-simple if all e_i's are distinct
- (vertex)-simple if all v_i's are distinct excepted maybe $v_0=v_\ell$.

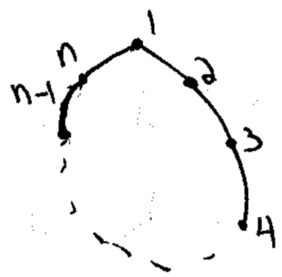
- The length is ℓ.

If $v_0=v_\ell$ we call it a cycle.

Def: If there is a chain between every pair of vertices in a graph $G=(V,E)$ we say that G is connected.

<u>Def</u>: The <u>girth</u> "g" of a graph is the length of the shortest cycle. If it is acyclic, it is defined as "∞".

<u>Examples</u>:

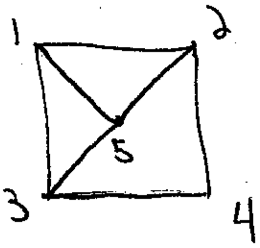
- The girth of the complete graph K_n, $n \geq 3$ is 3.
- The girth of the hypercube Q_n is 4.
- The <u>cycle graph</u> C_n: 

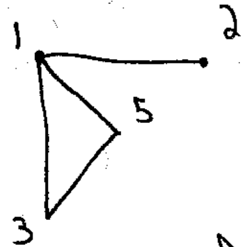
has girth n.

<u>Def</u>: Let $G = (V, E)$ be a graph. A <u>subgraph</u> $G' = (V', E')$ of G is such that $V' \subseteq V$ and $E' \subseteq E \cap (V' \times V')$.

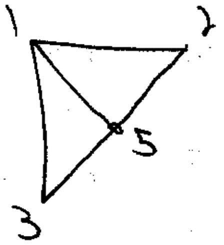
A subgraph is <u>induced</u> if $E' = E \cap (V' \times V')$.

<u>Example</u>:

G 

G_1 : 

G_1 is a subgraph of G
<u>not</u> induced
$\{2,5\}$ is missing.

G_2 : 

G_2 is an induced subgraph.

<u>Def:</u> A <u>connected component</u> of a graph G is an induced subgraph G' of G such that every pair of vertices in G' is connected by a chain in G' and every vertex of G connected by chain to a vertex of G' is in G'.

When G has only 1 connected component, G is connected.

<u>Def:</u> (Vertex-distance)

- Given two vertices $u, v \in V$, the <u>vertex-distance</u> $d(u,v)$ between u and v is the length of the shortest chain from u to v. If u and v are in 2 disjoint connected components then $d(u,v) = \infty$.

<u>Def:</u> (Diameter) The number $\max_{u,v \in V} d(u,v)$ is the <u>diameter</u> of the graph G.

<u>Example:</u> In the hypercube Q_n the distance between $u, v \in \{0,1\}^n$ is the number of bits on which they differ.

$\Rightarrow$ Diameter of $Q_n = n$.

<u>Thm:</u> A graph $G = (V, E)$ with $|V| \geq 2$ is bipartite $\Leftrightarrow$ all (simple) cycles have even lengths.

If they are no cycles (acyclic), then G is bipartite.

Proof: We may assume the graph to be connected. (Repeat for each component).

$\Rightarrow$] Every cycle must start in either U_1 or U_2 and come back ($V = U_1 \sqcup U_2$) Hence every cycle has even length.

$\Leftarrow$] Assume G has only even length cycles.

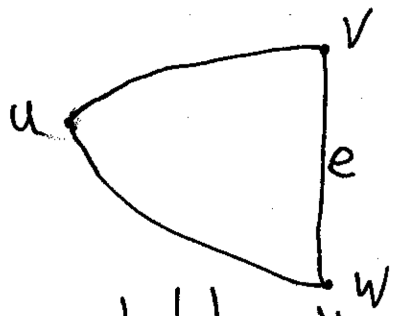
Pick $u \in V$ and define

$$U_0 = \{u\} \cup \{v \in V \mid d(u,v) \equiv 0 \mod 2\},$$
$$U_1 = \{v \in V \mid d(u,v) \equiv 1 \mod 2\}.$$

Clearly $V = U_0 \sqcup U_1$.

It remains to show that $E \subseteq U_0 \times U_1$.

Assume $e \in E$ and $e(v,w)$ w/ $v, w \in U_1$.



- $|d(u,v) - d(u,w)| \leq 1$

$\Rightarrow d(u,v) = d(u,w)$.

Conside two shortest path P_1: from u to v, P_2: from u to w and let x be their last common vertex,

then $d(x,v) = d(x,w)$

Then P_1, vw, P_2^{-1} is a cycle of odd length.
from w to x

$\lightning$.

The proof for U_0 is similar

$\boxtimes$

<u>Def:</u> Let G_1 and G_2 be two graphs. They are <u>isomorphic</u> if there exists a bijection

$$\varphi : V_1 \to V_2 \quad \text{such that}$$

$$(u,v) \in E_1 \iff (\varphi(u), \varphi(v)) \in E_2.$$

↳ · We can also define it for incidence structures
· We can also define an homomorphism. (more later).

- How many graphs are there on n vertices? $2^{\binom{n}{2}}$.
- How many graphs, up to isomorphism, are there on n vertices?
- How many connected graphs (up to isomorphism, or not) are there?
- How many graphs of girth 4 ... ?
- ...