Partially Ordered Sets

3. Lattices (continued)

Example: Young's lattice.

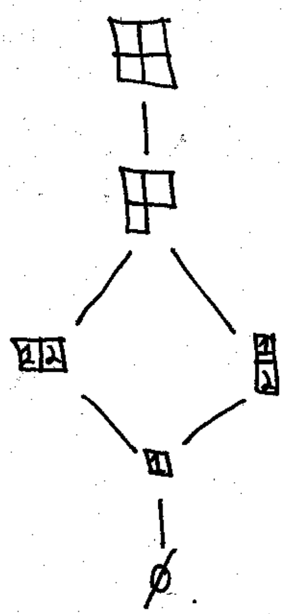
Let I be a principal lower ideal of $(Y, \subseteq)$; $I = \langle p \rangle$
$= \{\lambda \text{ part.} \mid \lambda \subseteq p\}$

The maximal chains from $\emptyset$ to p correspond to fillings of the boxes of the Ferrars diagram of p with numbers 1 to n where $\cdot\, n = \sum_{i=1}^{k} p_i$ $\quad p = p_1 + p_2 + \cdots + p_k$

- rows increase $\longrightarrow$
- colums increase $\downarrow$

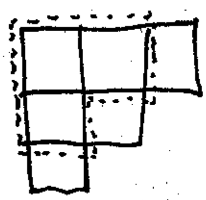
$$\underbrace{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}} \quad \longleftrightarrow$$

Young tableaux of shape $(2, 2)$.



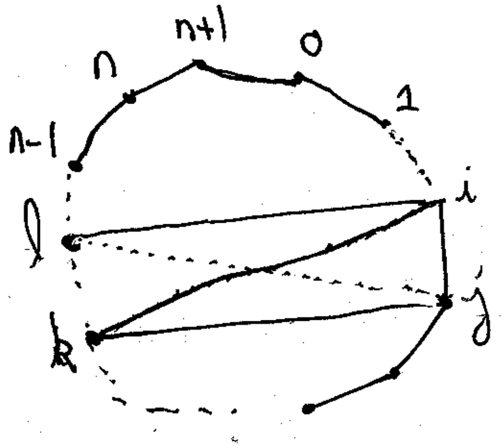
Similarly, maximal chains from p to q correspond to filling of skew shapes q/p.

$P = 2 + 1$
$Q = 3 + 2 + 1$



There are 6 fillings.

Example: Let $n \geq 1$, $T_n = \{$triangulations of a convex $(n+2)$-gon$\}$

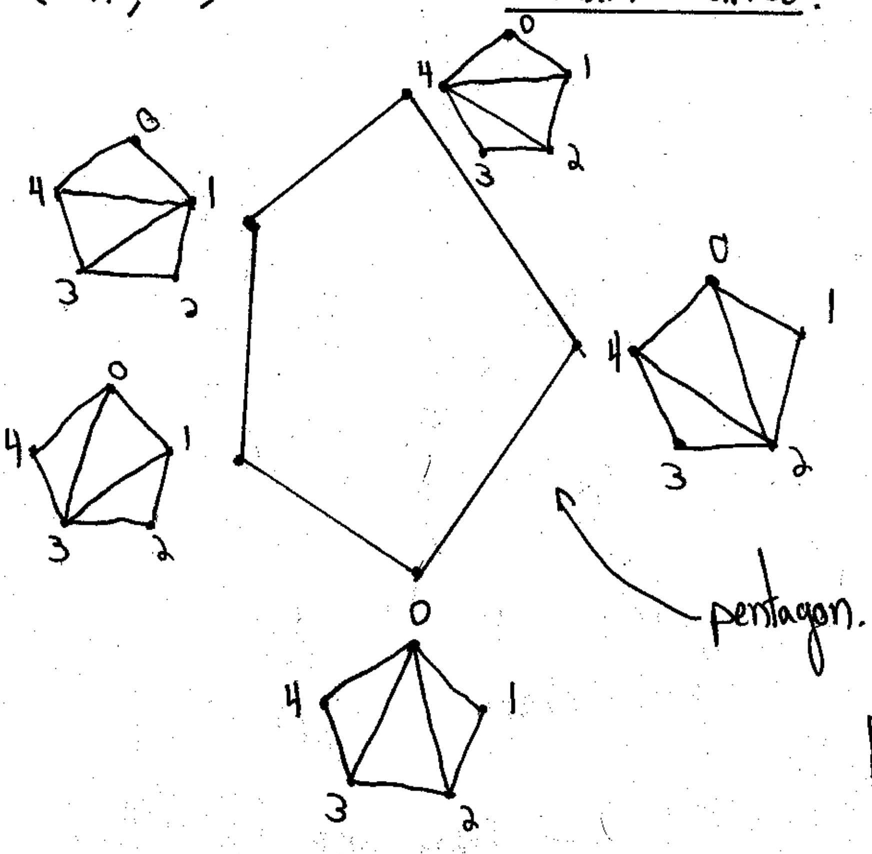


If $\{i,k\}$ is a diagonal, I can remove it and replace it with $\{j,l\}$ to get a new triangulation. This is an <u>increasing flip</u>.

t_1 covers t_2 $\iff$ t_1 has $\{i,k\}$, t_2 has $\{j,l\}$ and they share the other diagonals.

In general $t_1 \leq t_2$ if t_2 is obtained from t_1 by doing a sequence of increasing flips.

$(T_n, \leq)$ is the <u>Tamari lattice</u>.



<u>Fact:</u> The Hasse diagram of the Tamari lattice is the 1-skeleton, or graph, of a convex polytope called <u>Associahedron</u>

[1-skeleton: vertexe (0-dim) and edges (1-dim.) of the polytope.]

4. Incidence Algebra of a locally finite, poset.

Let P be a locally finite poset and let $\mathrm{Int}(P)$ be the set of intervals of P.

Consider the collection of functions from $\mathrm{Int}(P)$ to $\mathbb{C}$.
This is a vector space; • the zero-vector sends everything to 0.
• We can add and scale functions.

Let $\mathcal{I}(P)$ be this collection along with the convolution product:

$\mathbb{C}$-algebra.

$$(f * g)([x,y]) = \sum_{x \le z \le y} f([x,z]) \cdot g([z,y]).$$

- Since P is locally finite, this sum is finite.
- By abuse of notation, let $f([x,y]) = f(x,y)$ for $f \in \mathcal{I}(P)$.

- If P is finite, then we can choose a linear extension of P and represent $f \in \mathcal{I}(P)$ by the $|P| \times |P|$ matrix with rows and columns indexed by P and xy-entry to be $f(x,y)$.
When $x \not\le y$, then $f(x,y) = 0$, and such matrices are uppertriangular and convolution is matrix multiplication.

Some important functions:

The identity of $\mathcal{I}(P)$ is the delta function:

$\delta(x,y) = 1$ if $x = y$, and 0 otherwise.

The <u>zeta function</u> is

$$\zeta(x,y) = 1 \text{ if } x \leq y \text{ and } 0 \text{ otherwise.}$$

<u>Examples</u>:

1) Let P be an antichain of cardinality n.

Then $f(x,y)=0 \quad \forall\, x \neq y \in P$

What is δ? $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{n\times n}$.

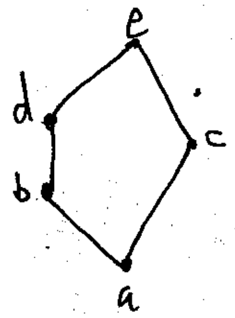
What is ζ? $\delta = \zeta$.

What is $\mathcal{I}(P)$? $\mathbb{C}^n$ (the diagonal $\mathbb{C}$-matrices).

2) Let P be a chain of cardinality n.

- What is δ? $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{n\times n}$
- What is ζ? $\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & 0 & 1 \end{pmatrix}_{n\times n}$
- What is $\mathcal{I}(P)$? The algebra of upper-triangular matrices.

3) Let P be



- What is δ? $\begin{pmatrix} 1 & & & & \\ & 1 & 0 & & \\ & & 1 & & \\ & 0 & & 1 & \\ & & & & 1 \end{pmatrix}$

- What is ζ? $\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ & & 1 & 0 & 1 \\ & 0 & & 1 & 1 \\ & & & & 1 \end{pmatrix} \end{array}$ $\rightarrow$ Zero's give incomparable elements.

<u>Observation:</u> If P is finite, the matrix of ζ is full rank. Thus ζ^{-1} exists.

In general, ζ always has an inverse!

<u>Definition:</u> The <u>Möbius function</u> $\mu \in \mathcal{I}(P)$ of a locally finite poset P is the convolution inverse of the zeta function ζ.

To show its existence, we give an explicit formula:

$$\mu(x,x) = 1 \quad \forall x \in P, \qquad \mu(x,y) = -\sum_{x \le z < y} \mu(x,z) = -\sum_{x < z \le y} \mu(z,y).$$

And check that:

$$\zeta * \mu = \mu * \zeta = \delta.$$

$$(\zeta * \mu)([x,y]) = \sum_{x \le z \le y} \underbrace{\zeta(x,z)}_{=1} \cdot \mu(z,y) = \sum_{x \le z \le y} \mu(z,y)$$

$$= 1 + \sum_{x \le z < y} \mu(z,y)$$

If $x = y$, then $(\zeta * \mu)(x,x) = 1$.

If $x < y$, We need

$$\boxed{0 = 1 + \sum_{x \le z < y} \mu(z,y)}$$

but

$$-\sum_{x < z \le y} \mu(z,y) \overset{\text{def}}{=} \mu(x,y)$$

$$-\sum_{x \le z \le y} \mu(z,y) = \mu(x,y) - \mu(x,y) = 0$$

$$-\sum_{x \le z < y} \mu(z,y) = \mu(y,y) = 1$$

Hence "=" holds.

For $(\mu * \zeta)$ it is similar.

<u>Lemma</u> Let P be a locally finite poset and $x < y$ be a cover of P.

Then $\mu(x,y) = -1$

Further if P is the chain $x_0 < x_1 < \dots < x_n$, then

$$\mu_P(x_0,x_0) = 1,\ \mu_P(x_0,x_1) = -1 \text{ and } \mu_P(x_0,x) = 0 \text{ if } x \neq x_0, x_1$$

<u>Proof:</u> Direct check.

Let $\mathcal{E}_{u,v}$ be the function $\mathcal{E}(u,v) = 1$ and 0 otherwise.

This is an <u>elementary matrix function</u>.

Lemma: If $f \in \mathcal{I}(P)$, then:

$$\varepsilon_{ux} * f * \varepsilon_{yv} = f(x,y) \cdot \varepsilon_{uv}$$

and

$$\varepsilon_{xx} * f * \varepsilon_{yy} = f(x,y) \cdot \varepsilon_{xy}$$

$\rightsquigarrow$ When u, v range over all pairs s.t. $u \leq v$ the functions ε_{uv} form a basis for $\mathcal{I}(P)$ as a $\mathbb{C}$-module.

Proposition: (Product Formula)
Let P and Q be partially ordered sets.
Then

$$\mu_{P\times Q}((x,u),(y,v)) = \mu_P(x,y) \cdot \mu_Q(u,v).$$

Proof: Check the recursion:

- If $(x,u) = (y,v)$ then $\mu_{P\times Q}((x,u),(y,v)) = 1 = 1 \cdot 1$ ✓
- Else

$$\sum_{(x,u)\leq(z,w)\leq(y,v)} \mu_{P\times Q}((x,u),(z,w)) = \sum_{x\leq z\leq y} \mu_P(x,z) \cdot \sum_{u\leq w\leq v} \mu_Q(u,w)$$

$$= \delta_{xy} \cdot \delta_{uv} = \delta_{(x,u),(y,v)}.$$

$\square$

Examples

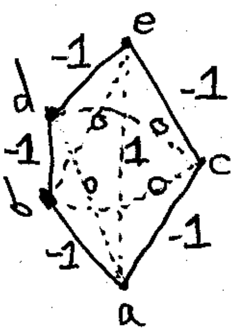
1) Let P be an antichain. Then $\delta = \zeta = \mu$.

2) Let P be a chain of cardinality n.
What is the inverse of $\zeta = \begin{pmatrix} 1 & \ddots & 1 \\ 0 & & 1 \end{pmatrix}$?

$$\mu = \begin{pmatrix} 1 & -1 & & 0 \\ & \ddots & \ddots & \\ & & & -1 \\ 0 & & & 1 \end{pmatrix}.$$ In general,

$\mu(x,y) = -1$ on covers
$\mu(x,x) = 1$
$\mu(x,y) = 0$ else.

3) Let P be

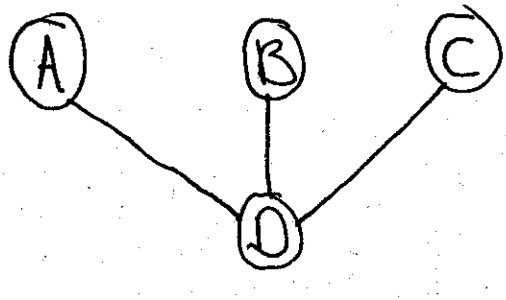


What is μ?

4) The Boolean lattice $(2^{[n]}, \subseteq)$ is the product $\begin{pmatrix} 1 \\ 0 \end{pmatrix}^n$
By the product formula; given $s, t \in 2^{[n]}$,

$$\mu(s,t) = (-1)^{|t|-|s|}.$$

5) Suppose you have 4 sets A, B, C and D such that:
$D = A \cap B = A \cap C = B \cap C = A \cap B \cap C$



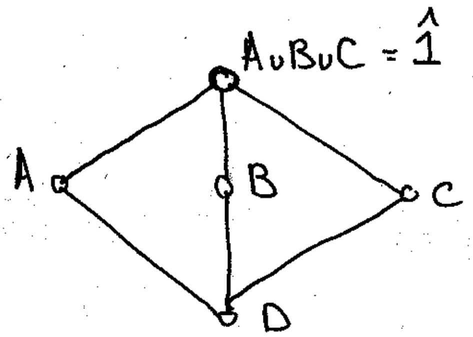
By the Inclusion Exclusion Principle:

$$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

$$= |A| + |B| + |C| - 2|D|$$

What does this mean?

Consider



What is μ?

$$\zeta = \begin{array}{c} \begin{array}{ccccc} D & A & B & C & \hat{1} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 1 \\ & & 1 & 0 & 1 \\ & 0 & & 1 & 1 \\ & & & & 1 \end{pmatrix} \end{array} \qquad \mu = \begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mu(D,\hat{1}) = 2, \quad \mu(A,\hat{1}) = \mu(B,\hat{1}) = \mu(C,\hat{1}) = -1.$$

<u>Definition:</u> A <u>multichain</u> is a multiset of elements $a_1, a_2, \dots, a_m$ satisfying $a_1 \leq a_2 \leq \dots \leq a_m$.

<u>Proposition:</u> Let $x \leq y \in P$ a locally finite poset. The number of multichains $x = x_0 \leq x_1 \dots \leq x_k = y$ is equal to $\zeta^k(x,y)$.

<u>Proof:</u> For $k=0,1$ it follows from the def. of δ and ζ.

By induction, assume it is true for values smaller than k.

Each multichain

$$\underbrace{x = x_0 \leq x_1 \leq x_2 \cdots \leq x_{k-1}}_{\text{length } k-1 \text{ multichain from } x \text{ to } z \in [x,y].} \leq x_k = y, \qquad x_{k-1} = z$$

$\oplus$ $\quad z \leq x_k = y$

By induction, $\cdot\ \zeta^{k-1}(x,z)$ is the number of m.chains from x to z.

$\cdot\ \zeta(z,y)$ is ——— " ——— z to y.

Summing $\forall z \in [x,y]$:

$$\sum_{z \in [x,y]} \zeta^{k-1}(x,z) \cdot \zeta(z,y) = \zeta^k(x,y).$$

<u>Proposition:</u> Let P be a locally finite poset. The number of chains of length k from x to y is

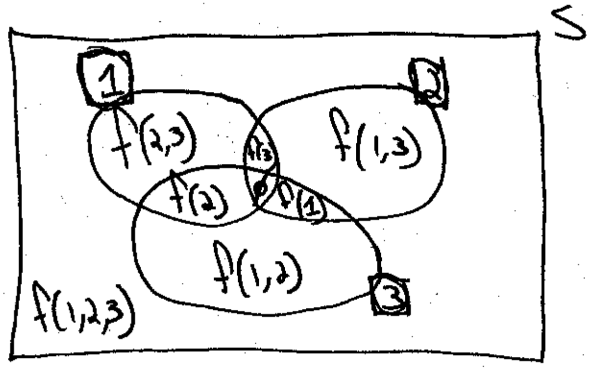
$$(\zeta - \delta)^k (x,y).$$

<u>Proof:</u> Similar to previous one. $\square$

Let S be a set and $A_i \subseteq S$, with $i \in [n]$.

Define $f(I) := \left| \bigcap_{i \notin I} A_i \setminus \bigcup_{i \in I} A_i \right|$ for $I \subseteq [n]$.

$$f(\emptyset) = \left| \bigcap_{i=1}^{n} A_i \right| \qquad f([n]) = \left| S \setminus \bigcup_{i=1}^{n} A_i \right|.$$



Then define $g(I) := \left| \bigcap_{i \notin I} A_i \right|$.

Then $g(I) = \sum_{J \subseteq I} f(J)$.

<u>Example:</u> $g(1,2) = f(\emptyset) + f(1) + f(2) + f(1,2) = |A_3|$

We can order S, A_i's and all intersections by inclusion...

The Inclusion-Exclusion Principle gives. (see Week 2):

$$f([n]) = \left| S \setminus \bigcup_{i=1}^{n} A_i \right| = |S| + \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right| = \sum_{I \subseteq [n]} \left| \bigcap_{i \notin I} A_i \right| \cdot (-1)^{n-|I|}$$

$$= \sum_{I \subseteq [n]} g(I) \cdot (-1)^{n-|I|}$$

Möbius for Boolean lattice!

Theorem: (Möbius Inversion Formula)

Let P be a poset in which each principal ideal is finite, and $f: P \to \mathbb{C}$. Farther let $g: P \to \mathbb{C}$ be

$$g(y) := \sum_{x \leq y} f(x) \quad \forall y \in P.$$

Then
$$f(x) = \sum_{y \leq x} \mu(y,x) \cdot g(y) \quad \forall x \in P.$$

And dually, if
$$g(y) := \sum_{y \leq x} f(x) \quad \forall y \in P,$$

then
$$f(x) = \sum_{x \leq y} \mu(x,y) \cdot g(y), \quad \forall x \in P.$$

Proof) We can rephrase:

$$g = f * \zeta \quad \Rightarrow \quad g * \mu = f$$

dually
$$g = \zeta * f \quad \Rightarrow \quad \mu * g = f.$$

Simply multiply by μ on both sides $\boxtimes$

Example:

- Take $(\mathbb{N}\setminus\{0\}, |)$.

  What is δ? $\delta : \mathbb{N}\setminus\{0\} \to \mathbb{C}$
  $$n \mapsto \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{else.} \end{cases}$$

  [Usually in number theory, they always look at intervals $[1,n]$.]

- What is ζ? $\zeta : \mathbb{N}\setminus\{0\} \to \mathbb{C}$
  $$n \mapsto 1$$

- What is μ?

$$\zeta * \mu = \delta$$

$$\sum_{d|n} \zeta(d) \cdot \mu\left(\frac{n}{d}\right) = \delta$$

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) = \delta$$

If $n=1$, $\mu(1) = 1$

Else
$$\sum_{d|n} \mu\left(\frac{n}{d}\right) = 0$$

$[1,n] \cong \bigotimes_{i=1}^{k} C_{a_i}$, where $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$
and C_{a_i} is a chain of length a_i

$$\Rightarrow \mu(n) = \begin{cases} (-1)^k & \text{if all } a_i \leq 1 \\ 0 & \text{else.} \end{cases}$$

→ See example on chains on page 8)

The number theoretic Möbius inversion formula is

$$f(n) = \sum_{d|n} g(d) \iff g(n) = \sum_{d|n} f(d)\,\mu(n/d)$$

$\forall n \geq 1$.

The theory of Möbius functions is <u>important</u> and has applications in

* Hopf algebras
* Ehrhart theory (order and chain polytopes
* Combinatorial Commutative algebra
* Number Theory
* Graph Theory (colorings of graphs)