Discrete Mathematics 1

Partially Ordered Sets

Motivation: Generalize the Inclusion-Exclusion Principle. and Möbius Inversion Formula in Number Theory:

If $F, G : \mathbb{N} \to \mathbb{C}$ such that $G(n) = \sum_{d|n} F(d) \quad \forall n \geq 1$

then $F(n) = \sum_{d|n} \mu(d) \cdot G(\frac{n}{d}) \quad \forall n \geq 1$

where μ is the Möbius function: $\mu(n) = 0$ if n is not sq. free.

$\mu(n) = (-1)^{\# \text{prime factors}}$

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1) Definitions & Examples

Def: An order relation $\leq$ on a set P is a binary relation on P for which:

- $x \leq x, \ \forall x \in P$ (reflexive)
- $x \leq y \ \& \ y \leq z \Rightarrow x \leq z \quad \forall x, y, z \in P$ (transitive)
- $x \leq y \ \& \ y \leq x \Rightarrow x = y \quad \forall x, y \in P$ (anti-symmetry)

The pair $(P, \leq)$ is called a partial ordered set (or poset).
If $x \leq y$, we say they are comparable.

Examples
- $(\mathbb{N} \setminus \{0\}, |)$
- $(\mathbb{Z}, \leq)$
- $(\mathbb{R}, \leq)$
- (Lin. subspaces of $\mathbb{R}^d$, $\subseteq$)
- $(2^{[n]}, \subseteq)$
- (Partitions, order $\lambda \leq \mu \overset{\text{def}}{\Leftrightarrow} \lambda_i \leq \mu_i \ \forall i$)
- (Partitions of n, dominance order)

- If $x \leqslant y$ or $y \leqslant x$ $\forall x, y \in P$, then "$\leqslant$" is a <u>total order</u> and P is <u>totally ordered</u>.
  - $(\mathbb{Z}, \leqslant)$ is totally ordered · $(\mathbb{N} \setminus \{0,1\}, |)$ is not.

- Two posets P, Q are <u>isomorphic</u> when there exists an <u>order preserving</u> bi-jection φ between P and Q:
$$\varphi : P \leftrightarrow Q \quad \text{s.t.} \quad \text{if } x \leqslant_P y \text{ with } x, y \in P$$
$$\text{then } \varphi(x) \leqslant_Q \varphi(y).$$

- An <u>interval</u> $[x, y]$ with $x \leqslant y \in S$ is the poset:
$$[x, y] = \{ z \in S \mid x \leqslant z \leqslant y \} \text{ with the order } \leqslant.$$

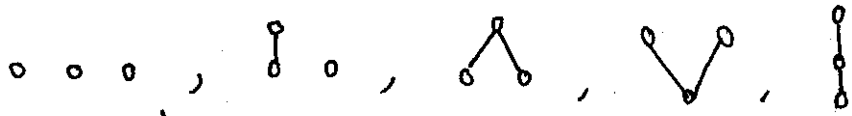
- If every interval in $(P, \leqslant)$ is finite, then $(P, \leqslant)$ is a <u>locally finite poset</u>.

<u>Example</u>: · $(\mathbb{N} \setminus \{0,1\}, |)$ and $(\mathbb{Z}, \leqslant)$ are locally finite.
· $(\mathbb{R}, \leqslant)$ is not.

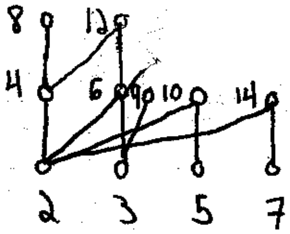
- If $[x, y] = \{x, y\}$, then y is a <u>cover</u> of x.

<u>Def</u>: The <u>Hasse diagram</u> of $(P, \leqslant)$ is a drawing (...a graph) where vertices are the elements of P and edges are cover relations and the cover is placed with higher "ordinate", "on top" of the element.

__Examples__ $|P| = 1$:  $|P| = 2$: 

$|P| = 3$: 

-  $\Rightarrow$ is not a Hasse diagram.

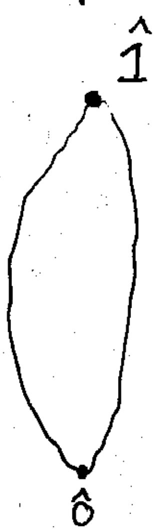
- $(\mathbb{N} \setminus \{0,1\}, \mid)$:  etc...

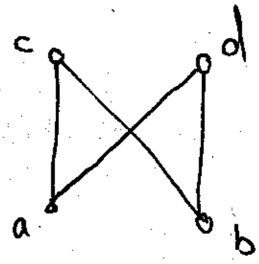
- We say P has a __greatest element__ $\hat{1} \in P$, if $x \leq \hat{1}$, $\forall x \in P$.
  ——— " ——— __least element__ $\hat{0} \in P$, if $\hat{0} \leq x$, $\forall x \in P$.

- An element $x \in P$ is a __maximal element__ if $\nexists\, a \in P$ such that $x \leq a$ and $a \neq x$.
  ——— " ——— __minimal element__ if $\nexists\, a \in P$ such that $a \leq x$ and $a \neq x$.

__Example__: • In $(\mathbb{N} \setminus \{0,1\}, \mid)$, there is no $\hat{0}$, $\hat{1}$ or maximal element. The minimal elements are prime numbers.

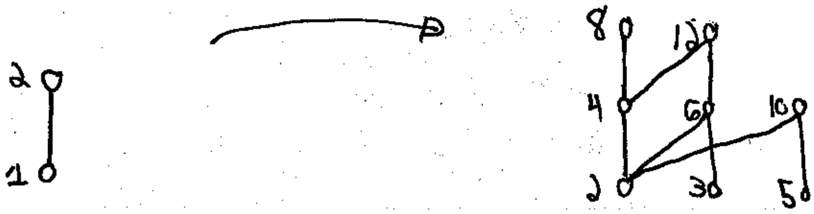
- In $(\mathbb{N} \setminus \{0\}, \mid)$ there is a least element $\hat{0} = 1$.
- In $([n], \subseteq)$ $[n] = \hat{1}$, $\emptyset = \hat{0}$.



 $\leftarrow$ Maximal
$\leftarrow$ Minimal

- A <u>chain</u> is the image of an order preserving map φ from $([n], \leq)$ to $(P, \leq)$, for $n \geq 1$.
  Since $([n], \leq)$ is totally ordered, <u>so is $\varphi([n])$</u>.

<u>Example</u>:
+(Example p8)



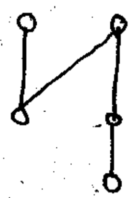
$\{2, 8\}, \{2, 4\}, \{2, 12\}, \ldots$ <u>They don't need to be covers!</u>
are chains of $(\mathbb{N} \setminus \{0\}, |)$.

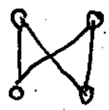
- The <u>length</u> of a chain is $n-1$. This is the number of 'up-steps'.

- The <u>rank</u> of a finite poset is the max. length of chains in it.
- If every maximal chain of P has the same length n, we say that P is <u>graded of rank n</u>.

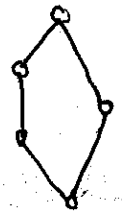
<u>Lemma:</u> If P is graded of rank n, then there exists a unique order preserving map ρ from $(P, \leq)$ to $(\{0, \ldots, n\}, \leq)$.
$\rho^{-1}(0)$ are the minimal elements.
$\rho^{-1}(n)$ are the maximal elements.
ρ is a <u>rank function.</u>

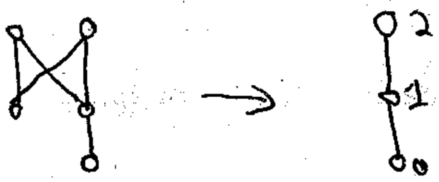
<u>Ex:</u>

 is not graded.

 is graded of rank 1.

- A finite poset is <u>ranked</u> if there exists a morphism $\rho : (P, \leq) \to ([n], \leq)$, where n is the rank of P, such that $\rho(u) = \rho(v) - 1$ whenever $u \lessdot v$ is a cover relation of P.

<u>Example</u>:  is <u>not</u> ranked.



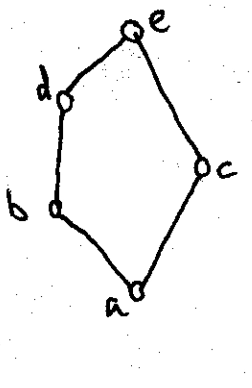
The <u>rank</u> of an element is then its image under ρ.

<u>Def</u>: An <u>antichain</u> of $(P, \leq)$ is a subset of elements of P that are pairwise incomparable.

<u>Example</u>: In $([5], \subseteq)$ $\{2,3\}$, $\{3,4\}$ are incomparable.

<u>Def</u>: An <u>order ideal</u> of $(P, \leq)$ is a subset I of elements of P that is "down-closed": if $x \in I$ and $y \leq x$ then $y \in I$.
An <u>order filter</u> is similar, but "up-closed".

<u>Ex</u>: What are the order ideals of ?



$\emptyset$, $\{a\}$, $\{a,b\}$, $\{a,c\}$, $\{a,b,c\}$, $\{a,b,d\}$, $\{a,b,c,d\}$, $\{a,b,c,d,e\}$

<u>Lemma</u>: If P is finite, then antichains of P are in bijection with order ideal of P.

<u>Proof</u>: Given an order ideal I since P is finite, I has a finite number of maximal elements M We can write

$$I = \{x \in P \mid \exists m \in M, \text{ s.t. } x \leq m\}.$$

But M is an antichain and it <u>uniquely describes</u> I. ☒.

Given a finite poset P, its order ideals are ordered by inclusion.

If $|M| = 1$, we say that I is <u>principal</u> generated by $m \in M$.

2) <u>Constructing posets</u>:

<u>Def</u>: The dual P^* of a poset P is $(P^*, \leq^*)$ where

$$x \leq^* y \overset{\text{def}}{\Longleftrightarrow} y \leq x.$$

$\rightarrow$ It reverses the order of P.

<u>Adding</u>: If P and Q are posets on disjoint sets, $P+Q$ is the poset on $P \cup Q$ s.t.

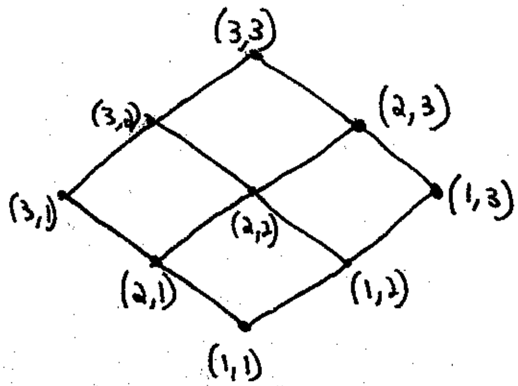
$$x \leq y \text{ in } P+Q \overset{\text{def}}{\Longleftrightarrow} \begin{cases} 1) & x, y \in P \text{ and } x \leq_P y \\ 2) & x, y \in Q \text{ and } x \leq_Q y. \end{cases}$$

<u>Multiplying</u>: If P and Q are posets then $P \times Q$ is the poset on the cartesian product "$P \times Q$" such that

$$(x, y) \leq (x', y') \text{ if } x \leq_P x' \text{ and } y \leq_Q y'.$$

$\rightarrow$ If $P \neq A + B$ for any A, B poset then P is <u>connected</u>.

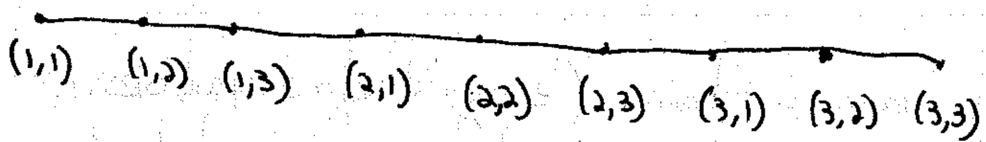
Example: • $([3] \times [3], \leqslant)$



In contrast:

• $([3] \times [3], \leqslant_{lex})$

$\rightarrow$



Def: Let A be a poset (alphabet), (usually totally ordered). The monoid A^* over A is ordered lexicographically by $\leqslant_{lex}$ as follows:

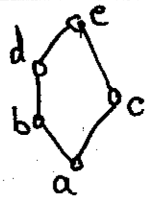
$w_1 < w_2 \overset{def}{\Longleftrightarrow}$
- w_1 is a prefix of w_2
- $\exists$ $u, v, w \in A^*$ (possibly empty) and $x < y \in A$ such that
  $w_1 = uxv$.
  $w_2 = uyw$.

Def: A well-order relation is a total order such that every non-empty subset has a least element.

Ex: Is $(\{a, b\}^*, \leqslant_{lex})$ well-ordered? No!

$\{a^n b \mid n \geqslant 1\}$ does not have a least element!

<u>Def</u>: A <u>linear extension</u> of a poset P is a morphism from P to $([n], \leq)$, where $n = |P|$.

<u>Ex</u>: What are the linear extensions of  ?

$[a, b, c, d, e]$

$[a, c, b, d, e]$ $\rightsquigarrow$ "Measure" the complexity of P.

$[a, b, d, c, e]$

Example: (Chains) How many maximal chains (image of covers are covers) are there in the Boolean poset $(2^{[n]}, \subseteq)$? and can not be extended to a longer chain

<u>Answer</u>: $n!$

A maximal chain looks like: $\emptyset \subset \{a\} \subset \{a,b\} \subset \cdots \subset [n]$

So, for each max. chain, write $\sigma = \begin{pmatrix} 1 & 2 & 3 & & n \\ a & b & \cdots & & \end{pmatrix}$

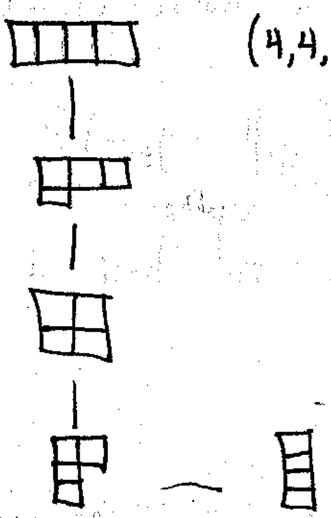
where $\sigma(i)$ is the i-th element added in the chain.

<u>Ex</u>: Fix $n \geq 1$. Take $P(n) = \{\text{partitions of } n\}$ and order them as follows:

$$\lambda \trianglelefteq \mu \iff \sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \quad \forall k \geq 1.$$

<u>Dominance order</u>

<u>$n=4$</u>



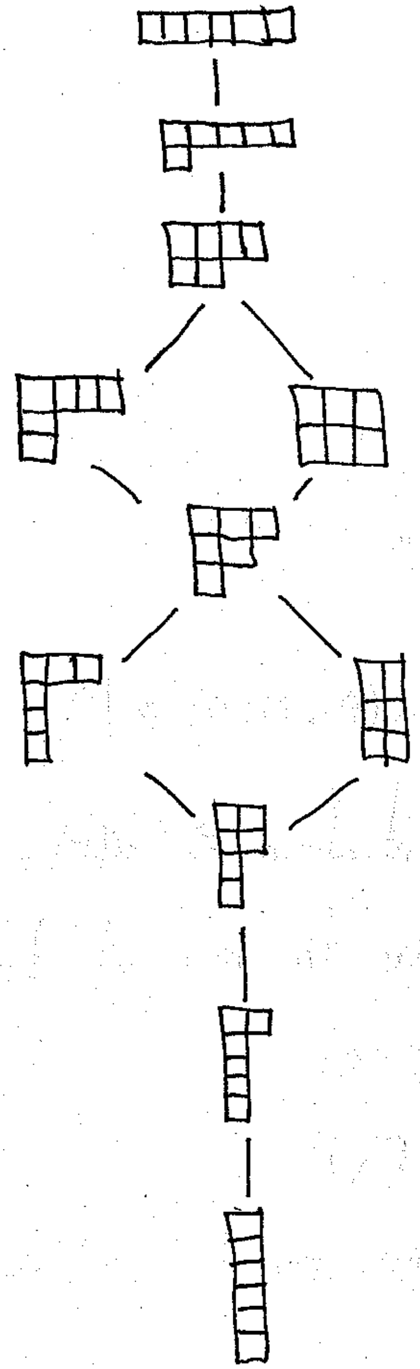
$(4, 4, 4, \ldots)$

$(3, 4, 4, \ldots)$

$(2, 4, 4, \ldots)$

$(2, 3, 4, 4 \ldots)$
$(1, 2, 3, 4$

$n=6$



Conjugating the partitions (NW-SE swap) is an order-reversing map of the dominance order poset.

$\hat{0} = (1, \dots, 1)$

$\hat{1} = (n)$

Theorem (Existence of linear extension)
Let P be a finite poset. There exists a linear extension of P,

pf By induction, if $|P|=1$, we are done as P is already a lin. order.

Claim: P has a minimal element x_0

pf of claim: To each $x \in P$, there are "m_x" $< \infty$ many element $y \in P$ such that $y \leq x$.

Pick x_0 "minimizing" this number m_x. (see a problem here?) assume Well-ordering for $\mathbb{N}$.

If $m_x = 1$, we are done x is a min. element.

Else take $y < x$, then $m_y < m_x$ ↯

By induction, get a linear extension of $P \setminus \{x_0\}$ into $[1, \dots, n]$ and send x_0 to 0. ☒

<u>Def</u>: • Let $\alpha(P)$ be the maximum size of an antichain in a poset P. (It may be infinite)

• Let $\beta(P)$ be the maximum size of a chain in a poset P. (The rank of $P+1$, it may be infinite).
($\beta(P)$ is the <u>height</u> of P)

<u>Thm</u>: Let P be a finite poset. Then $\alpha(P) \cdot \beta(P) \geq |P|$.

<u>Proof</u>: • If $\beta(P) = 1$, then P is an antichain $\Rightarrow \alpha(P) = |P|$ ✓

• By induction, if $\beta(P) > 1$, let M_1 be the minimal elements of P. M_1 is an antichain $\Rightarrow 1 \leq |M| \leq \alpha(P)$.
Let M_2 be the minimal elements of $P \setminus M_1$.
and so on with $M_3, M_4, \dots$ until we partition P completely.

$$P = M_1 \sqcup M_2 \sqcup \dots \sqcup M_t$$

But $t \leq \beta(P)$ because else we would have a longer chain ☒

<u>Corollary</u>: (Erdős-Szekeres Theorem)
An arbitrary sequence of n^2+1 real numbers contains a monotone subsequence of length $n+1$.

<u>Proof</u>: Let $\preceq$ be the order relation on $[n^2+1]$ defined by

$$i \preceq j \Leftrightarrow i \leq j \text{ and } s_i \leq s_j$$

where s_i is the i-th element in the sequence.

Claim: This is a poset. (Check this!)

$\overset{\text{Thm}}{\Rightarrow}$ $\alpha([n^2+1], \preccurlyeq) \cdot \beta([n^2+1], \preccurlyeq) \geq n^2+1$

$\Rightarrow$ $\alpha([n^2+1], \preccurlyeq) > n$ or $\beta([n^2+1], \preccurlyeq) > n$.

A chain in this order give an increasing subseq.
An antichain in this order gives a strictly decreasing subseq. $\boxtimes$

Corollary: (Mirsky's Theorem)
A poset P of height $\beta(P)$ can be partitioned into $\beta(P)$ antichains.

Proof: In the proof, $t = \beta(P)$. $\boxtimes$

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3. Lattices

Def.: • Given $S \subseteq P$, an upper bound of S is an element $z \in P$ s.t. $z \geq x$, $\forall x \in S$.

• A least upper bound (or "sup") of S is an upper bound z of S such that every upper bound y of S satisfies $y \geq z$.

We also say "join" of S and denote it $\bigvee_{s \in S} s$

or $x \vee y$ if $S = \{x, y\}$.

Think of "$\cup$".

If the join exists, it is unique (by def).

Similarly, define the lower bound, greatest lower bound, "inf" or meet, noted $\bigwedge_{s \in S} s$. (Think of "$\cap$").

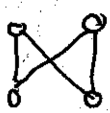
Def: • A lattice L is a poset such that every pair $x, y \in L$ has a join and a meet.

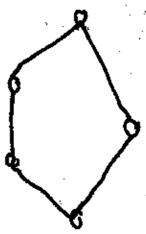
• A lattice is complete if the same is true $\forall S \subseteq L$.

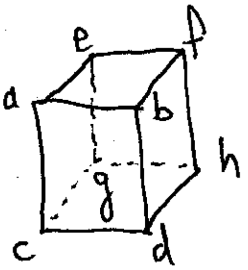
• A poset where the join of pairs exists is a join-semilattice.
Similarly for meet, we get a meet-semilattice.

Lemma: In a lattice L, the join $\vee$ and meet $\wedge$ operations are:

- associative
- commutative
- idempotent
- $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ (Absorbtion)
- $x \wedge y = x \iff x \vee y = y \iff x \leqslant y$.

Examples: →  is not a lattice.

→  is a lattice.

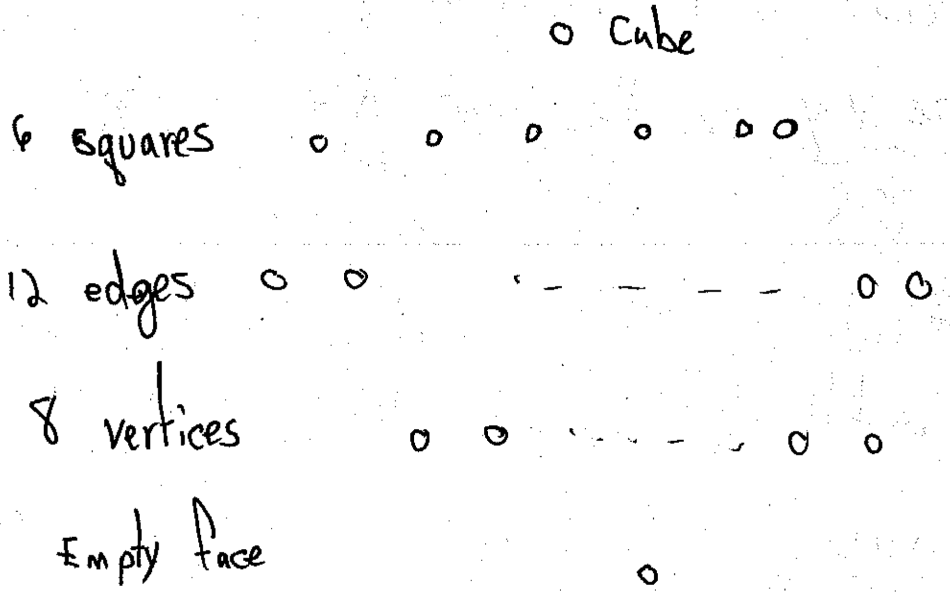


Faces of the cube ordered by inclusion is a lattice.

What is the meet? $\bigwedge_{f \in F} f = \bigcap$ of faces f.

F a set of faces.

$\bigvee_{f \in F} f$ = smallest face that contains all of them.



$(1, 6, 12, 8, 1)$
f-vector

Question: Characterize the f-vector of polytopes.

↳ This is a big driving research question in combinatorics of polytopes.

Proposition: (Criterions to get a finite lattice)
Let P be a finite meet-semilattice with $\hat{1}$.
Then P is a lattice.
Dually, a finite join-semilattice with $\hat{0}$ is a lattice.

<u>Pf</u>] The intersection F of the filters $\{z \in P \mid z \geqslant x\}$ and $\{z \in P \mid z \geqslant y\}$ is <u>finite</u> and <u>non-empty</u> because $\hat{1}$ is in both.

By induction: F is a smaller meet-semilattice with $\hat{1}$

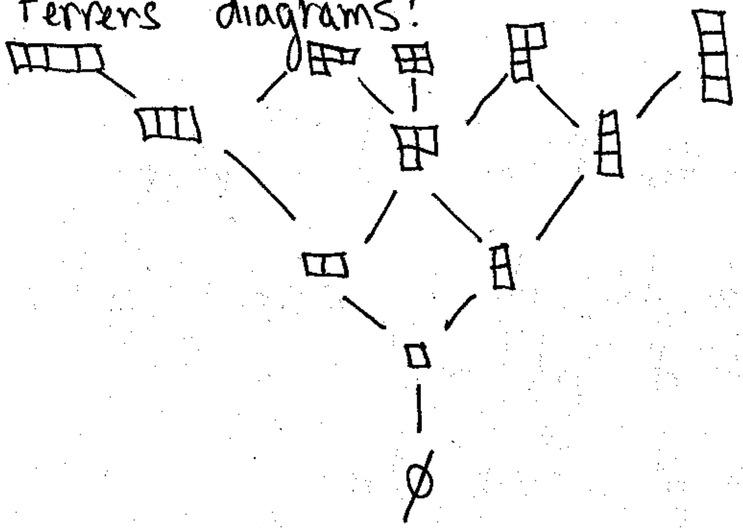
$$\Downarrow$$

the meet of elements in F excists "$x \wedge y$" $\in F$

So define $\underbrace{x \vee y}_{\text{join}}$ as the meet $\bigwedge_{z \in F} z$. $\boxtimes$

<u>Example</u>: (<u>Young's lattice</u>)

Let $Y = \{$ integer partitions $\}$ and order Y by inclusion of their Ferrers diagrams:



- It is <u>graded</u>.
- <u>Meet</u>: intersection of diagrams
- <u>Join</u>: union of diagrams

$\to$ Representations of S_n and branchings $\to$ Important in Algebraic Combinatorics.