Discrete Mathematics 1

Applications of Generating Functions

Example 1: Determine the number of bags of fruit made of apples, bananas, oranges and pears where, in each bag:

- the number of apples is even,
- —— " —— bananas is a multiple of 5,
- —— " —— oranges is at most 4,
- —— " —— pears is 0 or 1.

Solution: Let B_n be the number of bags with n fruits.
We determine the generating function for B_n:

$$B(x) = \underbrace{(1 + x^2 + x^4 + \dots)}_{\text{Choosing apples}} \cdot \underbrace{(1 + x^5 + x^{10} + \dots)}_{\text{Choosing bananas}} \cdot \underbrace{(1 + x + x^2 + x^3 + x^4)}_{\text{Choosing oranges}}$$

$$\cdot \underbrace{(1 + x)}_{\text{Choosing pear.}}$$

Each factor corresp. to a choice of fruit.

$$B(x) = \frac{1}{1-x^2} \cdot \frac{1}{1-x^5} \cdot \frac{1-x^5}{1-x} \cdot (1+x) = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} \binom{n+1}{n} x^n.$$

$$= \sum_{n=0}^{\infty} (n+1)\, x^n.$$

Thus, to form a bag with n fruits there are $n+1$ ways. ☒

Notice how we barely did counting, but merely algebraic manipulations.

Binomial Theorem for negative exponents

From the equation

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \in \mathbb{C}[[x]],$$

we can look at $\frac{1}{(1+x)^m} \in \mathbb{C}[[x]]$.

What is the coefficient of x^n in this FPS?

Thm: $$\left[\frac{1}{(1+x)^m}\right]_{x^n} = (-1)^n \binom{m+n-1}{n}.$$

Pf] Consider $\frac{1}{1-x} = 1 + x + x^2 + \dots$.

The coefficient of x^n in $\frac{1}{(1-x)^m}$ is equal to the number of ways to put "n" exponents (unlabeled) into "m" terms (labeled)

→ This is the number of weak compositions of n into m blocks.

→ By the T.W. this is

$$\binom{n+m-1}{m-1} = \binom{n+m-1}{n}.$$

Replacing x by "$-x$" we get the result. ☒

Therefore we get the extension: $\alpha \in \mathbb{Z}$, $n \in \mathbb{N}$

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}, \quad \text{with } \binom{\alpha}{0} = 1$$

If $\underset{-m}{\overset{\|}{\alpha}} < 0$, $$\binom{\alpha}{n} = \binom{-m}{n} = (-1)^n \frac{m(m+1)(m+2)\cdots(m+n-1)}{n!} = (-1)^n \binom{m+n-1}{n}.$$

Solve Recurrence Relations via GFs

Def: Let $F(x) \in \mathbb{C}[[x]]$. $F(x)$ is a rational fct in $\mathbb{C}[[x]]$ when there exist $P(x), Q(x) \in \mathbb{C}[[x]]$ s.t. $F(x) = P(x) \cdot Q(x)^{-1}$. ($Q(0) \neq 0$).

Thm: Let $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{C}$, $k \geq 1$ and $\alpha_k \neq 0$. The FAE on $S: \mathbb{N} \to \mathbb{C}$

i) $\displaystyle\sum_{n \geq 0} S(n) x^n = \frac{P(x)}{Q(x)}$, where $Q(x) = 1 + \alpha_1 x + \dots + \alpha_k x^k$ and $\deg(P) < k$.
polynomial.

ii) For all $n \geq 0$,

$$S(n+k) + \alpha_1 S(n+k-1) + \dots + \alpha_k S(n) = 0$$

iii) For all $n \geq 0$,

$$S(n) = \sum_{i=1}^{k} P_i(n) \gamma_i^n, \quad \text{where } 1 + \alpha_1 x + \dots + \alpha_k x^k = \prod_{i=1}^{j} (1 - \gamma_i x)^{d_i}$$

the γ_i's are distinct and $P_i(n)$ is a polynomial in n of degree $< d_i$

Pf Let $V_1 = \{ S : \mathbb{N} \to \mathbb{C} \text{ s.t. (i) holds} \}$

$V_2 = \{ \text{ — " — (ii) holds} \}$

$V_3 = \{ \text{ — " — (iii) holds} \}$

$V_4 = \{ S : \mathbb{N} \to \mathbb{C} \text{ s.t. } \sum S(n) x^n = \sum_{i=1}^{j} G_i(x) (1 - \gamma_i x)^{-k_i} \text{ as in (iii)} \}$

V_1, V_2, V_3, V_4 all have dimension k.

Fix $Q(x) = 1 + \alpha_1 x + \dots + \alpha_k x^k$.

- If $S \in V_1$, then $Q(x) \cdot \sum_{n \geq 0} S(n) x^n = P(x)$.

  and the coeff. of x^n is (when $n \geq k$)

$$S(n) + \alpha_1 S(n-1) + \alpha_2 S(n-2) + \dots + \alpha_k S(n-k) = 0$$

$\Rightarrow S \in V_2$. Since $\dim V_1 = \dim V_2 \Rightarrow V_1 = V_2$.

- If $S \in V_4$, express it as a fraction, $S \in V_1$ since $\dim V_1 = \dim V_4$

  $\Rightarrow V_1 = V_4 = V_2$.

- $\sum_{i=1}^{j} G_i(x)(1-\gamma_j x)^{-k_i}$ is a fin. lin. comb of $x^l (1-\gamma x)^{-c}$ where $l < c$.

$$\frac{x^l}{(1-\gamma x)^c} = x^l \sum_{n \geq 0} (-\gamma)^n \binom{-c}{n} x^n = \sum_{n \geq 0} x^n \gamma^n \gamma^{-l} \binom{c+n-1-l}{c-1}.$$

Since $\gamma^{-l}\binom{c+n-1-l}{c-1}$ is a polynomial in n of deg $c-1$,

$\Rightarrow V_4 \subseteq V_3$. $\oplus$ $\dim V_3 = \dim V_4 \Rightarrow V_3 = V_4$. $\boxtimes$

<u>Ex</u> 2) $P(n)$ = # paths from $(0,0)$ using $W = (-1,0)$, $E = (1,0)$, $N = (0,1)$

that do not intersect themselves.

$\Leftrightarrow$ # words of length n such that "WE" or "EW" are never factors.

Let $n \geq 2$, • there are $P(n-1)$ words ending in "N",

• there are $P(n-1)$ words — " — "EE", "WW" or "NE",

• — " — $P(n-2)$ — " — "NW",

and they are all ending in the above way.

Hence $P(n) = 2P(n-1) + P(n-2)$, $P(0) = 1$, $P(1) = 3$.

By the <u>Thm</u>: $\exists\, A, B \in \mathbb{R}$ s.t.

$$\sum_{n \geq 0} P(n) x^n = \frac{A + Bx}{1 - 2x - x^2}$$

Reflected. Char. poly.

To get A, B, multiply both sides by $1 - 2x - x^2$.

$\Rightarrow A = B = 1$.

$$\sum_{n \geq 0} P(n) x^n = \frac{1 + x}{1 - 2x - x^2}$$

<u>Partial fractions</u>: $\dfrac{1 + x}{1 - 2x - x^2} = \dfrac{-1/2}{(x - \sqrt{2} + 1)} + \dfrac{-1/2}{(x + \sqrt{2} + 1)}$

Part iii): $P(n) = \dfrac{1}{2}\left[(1 + \sqrt{2})^{n+1} + (1 - \sqrt{2})^{n+1}\right]$.

There are 3^n paths in total

By restricting, we get roughly $(2{,}414\ldots)^n$ paths.

<u>Corollary</u>: Let $S: \mathbb{N} \to \mathbb{C}$ and $k \in \mathbb{N}$. TFAE:

i) $\displaystyle\sum_{n \geq 0} S(n) x^n = \frac{P(x)}{(1 - x)^{k+1}}$, where $P(x) \in \mathbb{C}[x]$ and $\deg P \leq k$.

ii) $\forall n \geq 0$, $\displaystyle\sum_{i=0}^{k+1} (-1)^{k+1-i} \binom{k+1}{i} S(n+i) = 0$

iii) $S(n)$ is a polynomial fct of n of deg. at most k.

Ex3: (Fibonacci again)

Step 1) Express the RR in term of the GF.

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad (\forall n \geq 2)$$

That means
$$F(x) = \sum_{n\geq 0} F_n x^n = \sum_{n \geq 2} (F_{n-1} + F_{n-2}) x^n + x$$

$$= x + \sum_{n\geq 2} F_{n-1} x^n + \sum_{n \geq 2} F_{n-2} x^n$$

$$= x + \sum_{n \geq 1} F_n x^{n+1} + \sum_{n\geq 0} F_n x^{n+2} = x + x F(x) + x^2 F(x)$$

$$\iff -x^2 F(x) - (x-1) F(x) - x = 0$$

$$\iff \boxed{F(x) = \frac{x}{-x^2 - x + 1}}$$

Step 2) Partial Fractions:
$$F(x) = x\left[\frac{A}{1-\gamma_1 x} + \frac{B}{1-\gamma_2 x}\right]$$

where $\gamma_1 = \frac{+1+\sqrt{5}}{2}$, $\gamma_2 = \frac{+1-\sqrt{5}}{2}$, $A = \frac{\gamma_1}{\sqrt{5}}$, $B = \frac{-\gamma_2}{\sqrt{5}}$

$$F(x) = \left[\underbrace{\frac{1}{\sqrt{5}} \cdot \gamma_1^n} - \frac{1}{\sqrt{5}} \cdot \gamma_2^n\right]$$

Roots of the reflected polynomial.
(The char. poly.)

because $\frac{1}{1-\gamma_1 x} = 1 + \gamma_1 x + \gamma_1^2 x^2 + \gamma_1^3 x^3 + \ldots$

Ex. 4: In how many ways can we form units (non-empty) that each have a captain out of "n" aligned soldiers?

Solution:
Step 1) • Choosing a captain in a group of "k" people is done in "k" ways:

$$\mathcal{A}(x) = \sum_{k\geq 0} k\, x^k = \frac{x}{(1-x)^2} \quad \text{(Check!)}.$$

• By the A.P. either I form 1, 2, …, teams and then choose a captain for each of them.
Let $\mathcal{H}(x)$ be the G.F. for our problem, then

$$\mathcal{H}(x) = \underset{\substack{\uparrow \\ \text{on the} \\ \text{empty set} \\ \text{we do "nothing"}}}{1} + \underset{\substack{\uparrow \\ 1 \text{ team}}}{\mathcal{A}(x)} + \underset{\substack{\uparrow \\ 2 \text{ team}}}{\mathcal{A}^2(x)} + \cdots$$

$$\underset{\substack{\uparrow \\ \text{Since} \\ \mathcal{A}(0)=0}}{=} \frac{1}{1-\mathcal{A}(x)} = \mathcal{E}(\mathcal{A}(x)) \leftarrow \text{"Sets of teams w/captain" (labeled)}$$

$$= \frac{1}{1-\frac{x}{(1-x)^2}} = 1 + \frac{x}{1-3x+x^2}.$$

Step 2) $\dfrac{1}{1-3x+x^2} \Rightarrow$ roots of x^2-3x+1 are $\alpha = (3+\sqrt{5})/2$, $\beta = (3-\sqrt{5})/2$

$$= \frac{1}{(x-\alpha)(x-\beta)} = \frac{A}{x-\alpha} - \frac{B}{x-\beta}$$

$$\Rightarrow A = B = 1/\sqrt{5}.$$

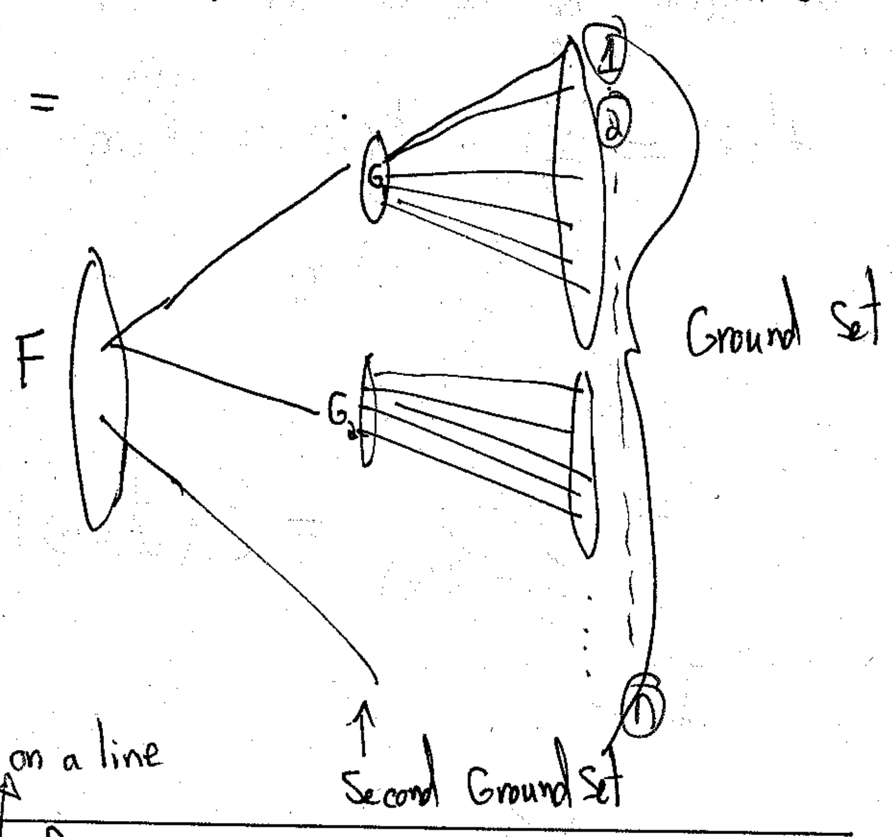
$$\frac{1}{1-3x+x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{x-\alpha} - \frac{1}{x-\beta}\right) \underset{\alpha\cdot\beta=1}{=} \frac{1}{\sqrt{5}}\left(\frac{\alpha}{1-\alpha x} - \frac{\beta}{1-\beta x}\right).$$

$$\hookrightarrow \left[\frac{1}{1-3x+x^2}\right]_{x^n} = \frac{1}{\sqrt{5}}\left(\alpha^{n+1} - \beta^{n+1}\right)$$

$\Rightarrow$ $H(0)=0$ and $n \geq 1$ $h(n) = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right)$ $\boxtimes$

<u>Composition</u>: First construct G-structures on non-empty intervals then on each of these G-structure, construct a F-structure:

$F(G(x)) =$



<u>Exc 5</u>: Given "n" people, form (non-empty) teams and then choose some teams (maybe none) to work on a project. In how many ways can this be done?

<u>Solution</u>: $A(x) \rightarrow$ "Forming non-empty sets"$(x) = \frac{x}{1-x}$ (1 way for each cardinality >0).

$B(x) \rightarrow$ "Forming a subsets of n elements"$(x) = \frac{1}{1-2x}$

Let $G(x)$ be the G.F. counting the # of ways to "Form non-empty sets and then a subset of these from $[n]$".

$$\Rightarrow G(x) = \mathcal{B}(\mathcal{A}(x)) = \frac{1}{1-\frac{2x}{1-x}} = \frac{1-x}{1-3x} = \frac{1}{1-3x} - \frac{x}{1-3x}.$$

$$= \sum_{n\geq 0} 3^n x^n - \sum_{n\geq 1} 3^{n-1} x^n = 1 + \sum_{n\geq 1} 2\cdot 3^{n-1}.$$

If $n\geq 1$, there are $2\cdot 3^{n-1}$ ways.

Ex.6: In how many ways can you select some books from a series of n books and then pick your favorite two?

Solution:

- Say you have a GF $F(x)$, tagging one element in $F(n)$ is given by $\underset{\text{"puts it back w/ a label"}}{x}\cdot\underbrace{F'(x)}_{\text{"removes an element"}}$.

- We want to pick twice without putting back.

$$\Rightarrow \quad x^2\cdot F''(x) = x^2\cdot\sum_{n\geq 2} n(n-1)F_n\cdot x^{n-2} = \sum_{n\geq 0} n(n-1)F_n x^n$$

In our case $F(x) = \dfrac{1}{1-2x} = 1+2x+4x^2+8x^3\dots$

$$x^2\cdot F''(x) = \sum_{n\geq 0} n(n-1)\cdot 2^n\cdot x^n.$$

Answer: $\boxed{n(n-1)\cdot 2^n.}$

Catalan Numbers

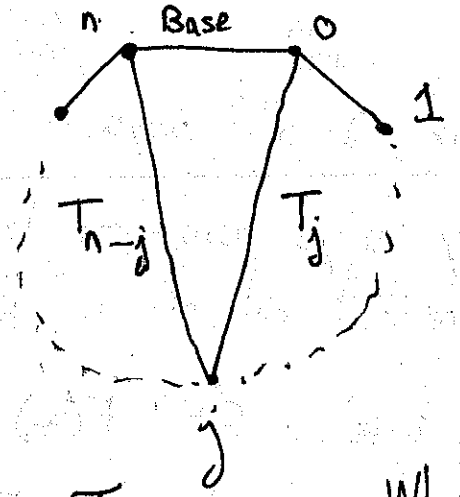
Let T_n be the number of ways to triangulate a convex $(n+1)$-gon : (add diagonals that do not intersect in their interior until there are only Δ's).

$T_3 = 2$, $T_4 = 5$, $T_5 = 14$, $T_6 = 42$

$\boxed{T_2 = 1}$

$\boxed{T_1 = 1}$

$T_0 = 0$



$$T_n = \sum_{j=1}^{n-1} T_{n-j} \cdot T_j$$

What is T_n ?

Thm: $T_n = \dfrac{1}{n}\dbinom{2(n-1)}{n-1}$ $(n \geq 1)$.

pf| Consider $T(x)$, the GF for T_n.

Then
$$T^2(x) = \sum_{n\geq 2}^{\infty}\left(\sum_{i=0}^{n} T_i \cdot T_{n-i}\right)x^n$$

$$= \sum_{n\geq 2}^{\infty} T_n \cdot x^n = T(x) - T_1 x = T(x) - x$$

$$\Leftrightarrow T^2(x) - T(x) + x = 0$$

$\Leftrightarrow \quad T(x) = \frac{1+\sqrt{1-4x}}{2}$ or $= \frac{1-\sqrt{1-4x}}{2}$

Since $T(0)=0$, we take

$$T(x) = \frac{1-\sqrt{1-4x}}{2} = \frac{1}{2} - \frac{1}{2}(1-4x)^{1/2}$$

Take the Bin. Thm w/ real values:

$$(1+x)^{1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 2^{2n-1}} \binom{2n-2}{n-1} x^n, \quad (|x|<1).$$

↳ Substitute x for $-4x$:

$$(1-4x)^{1/2} = 1 - 2\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^n, \quad (|x| < 1/4).$$

↳ Thus $T(x) = \frac{1}{2} - \frac{1}{2}(1-4x)^{1/2} = \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^n.$

$$\Rightarrow T_n = \frac{1}{n}\binom{2(n-1)}{n-1}. \quad n \geq 1 \quad \boxtimes$$

Exponential Generating Functions

So far, the GF were build for labeled (ordered.) sets, what if we build structures on unlabeled sets?

↳ Exponential GFs!

↳ If terms count things related to permutations...

Def: Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.
The FPS $\mathcal{S}^{(e)}(x) := \sum_{n\geq 0} S_n \cdot \frac{x^n}{n!}$ is called the <u>exponential generating function</u> of $(S_n)_{n\in\mathbb{N}}$.

<u>Ex.</u> • $(n!)_{n\in\mathbb{N}} \rightarrow \sum_{n\geq 0} x^n = \frac{1}{1-x}$ is the exp. GF of permutations!

• $(1)_{n\in\mathbb{N}} \rightarrow \mathcal{E}^{(e)}(x) = \sum_{n\geq 0} \frac{x^n}{n!} = e^x$

• $((n-1)!)_{n\in\mathbb{N}} \rightarrow \sum_{n\geq 1} \frac{(n-1)!}{n!} x^n = \sum_{n\geq 1} \frac{x^n}{n} = \log\left(\frac{1}{1-z}\right)$

<u>RR and EGFs:</u>

<u>Ex:</u> $S_0 = 0, \quad S_{n+1} = 2(n+1)S_n + (n+1)!$

<u>Solution:</u> $\mathcal{S}^{(e)}(x) = \sum_{n\geq 0} S_n \frac{x^n}{n!}$

• Multiply both sides by $\frac{x^{n+1}}{(n+1)!}$ and sum $\forall n \geq 0$.

$$\sum_{n\geq 0} S_{n+1} \frac{x^{n+1}}{(n+1)!} = 2x \sum_{n\geq 0} S_n \frac{x^n}{n!} + \underbrace{\sum_{n\geq 0} x^{n+1}}_{\frac{x}{1-x}}$$

Since $S_0 = 0$, $LHS = \mathcal{S}^{(e)}(x)$

$2x \sum_{n\geq 0} S_n \frac{x^n}{n!} = 2x \mathcal{S}^{(e)}(x)$

Hence $S^{(e)'}(x) = 2x\, S^{(e)}(x) + \frac{x}{1-x}$.

$\iff S^{(e)}(x) = \frac{x}{(1-x)(1-2x)} = \sum_{n\geq 0} (2^n - 1)\, x^n$

$\iff S_n = (2^n - 1)\cdot n!$

<u>Ex:</u> <u>Bell numbers</u>

We know that $B(n+1) = \sum_{i=0}^{n} B(i)\binom{n}{i}$ $\forall n \geq 0$ and $B(0) = 1$.

· Multiplying both sides by $\frac{x^n}{n!}$ and sum $\forall n \geq 0$:

$$\underbrace{\sum_{n\geq 0} B(n+1)\frac{x^n}{n!}}_{B^{(e)'}(x)} = \underbrace{\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} B(i)\binom{n}{i}\right)\frac{x^n}{n!}}_{B^{(e)}(x)\cdot e^x}.$$

$$B^{(e)'}(x) = B^{(e)}(x)\cdot e^x \iff \frac{B^{(e)'}(x)}{B^{(e)}(x)} = e^x$$

Integrating:

$$\log B^{(e)}(x) = e^x + c.$$

Since $B^{(e)}(0) = 1$, $\Rightarrow$ ~~c=1~~. $C = -1$

$$\iff \log B^{(e)}(x) = e^x - 1.$$

$$\iff \boxed{B^{(e)}(x) = e^{e^x - 1}}$$

A set-partition is a set of non-empty sets.
that means: $e^{e^x - 1}$.

Lemma: $(A_n)_{n\geq 0}$ $(B)_{n\geq 0}$

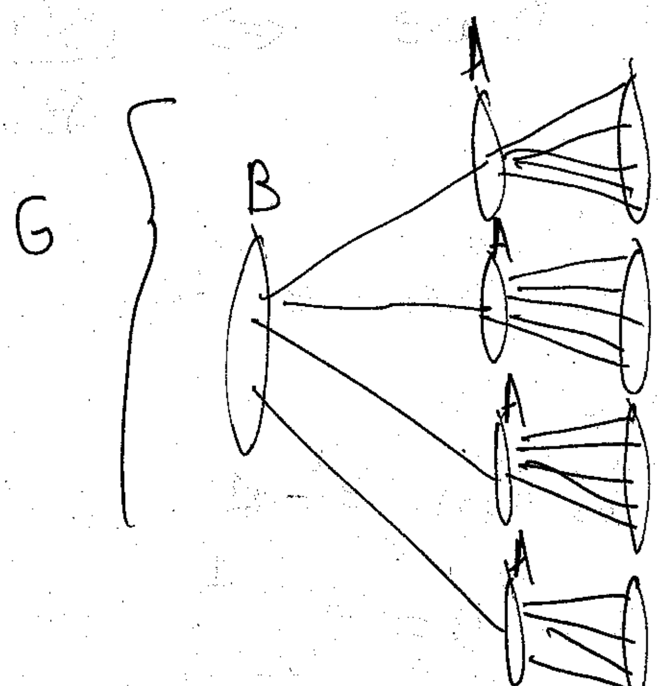
$$\mathcal{A}^{(e)}(x) \cdot \mathcal{B}^{(e)}(x) = \sum_{n\geq 0} c_n \frac{x^n}{n!}, \quad w/ \ c_n = \sum_{i=0}^{n} \binom{n}{i} A_i \cdot B_{n-i}.$$

Product formula: This count the # of ways to split $[n]$ into two subsets (not necessarily initial/final) and build A on the first and B on the second.

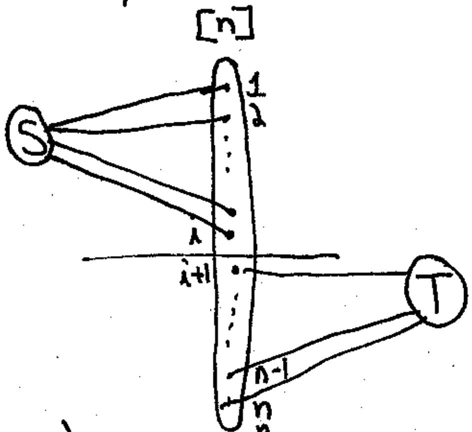
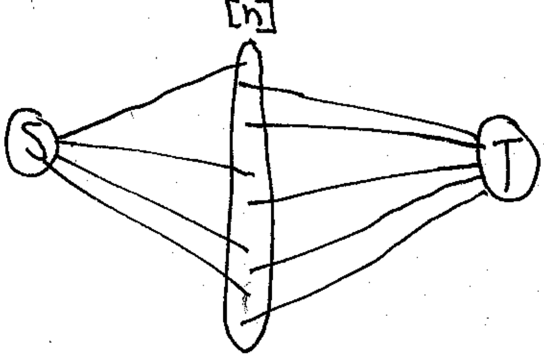
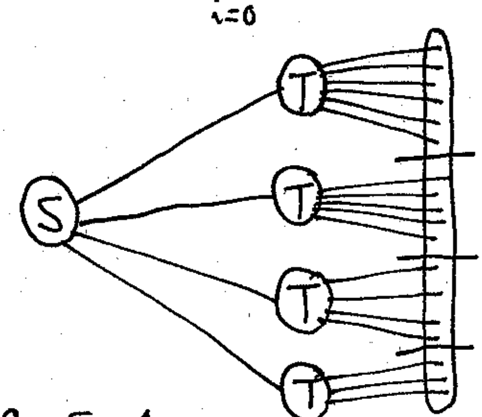
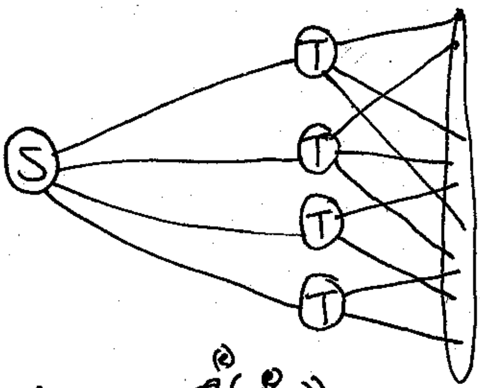
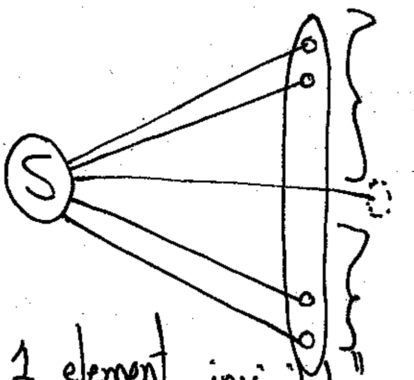
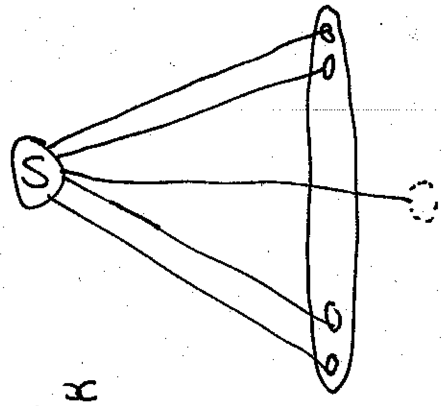
Composition formula: Again the "first" should not allow empty set construction. If the "second" is to build a set:

$$\Rightarrow \quad e^{A(x)}.$$

In general $\mathcal{G}(x) = \mathcal{B}(\mathcal{A}(x))$



(Good for combinations) (Good for permutations).

	Ordinary Gen. Funct.	Exponential Gen. Funct.
Product		
	$(S\cdot T)_{n\in\mathbb{N}} = \sum_{i=0}^{n} S_i \cdot T_{n-i}$	$(S^{(e)}\cdot T^{(e)})_{n\in\mathbb{N}} = \sum_{i=0}^{n} \binom{n}{i} S_i \cdot T_{n-i}$
Composition		
	$T_0=0$ $S_0=1$ $S(T(x))$	$T_0=0$, $S_0=1$ $S^{(e)}(T^{(e)}(x))$
Derivative		
	"Make 1 element invisible"	$(e^{x})' = e^{x}$
	$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = \sum_{n\geq 0}(n+1)x^n$	When removing an element (not labeled!) from a set, we get a set back.

Ways to split [n] into two intervals

Ordinary Generating Fcts

Ground set: "Unlabeled" $\downarrow$ Hence we may fix a canonical order.

Product: $(S \cdot T)_{n \in \mathbb{N}} = \left(\sum_{i=0}^{n} S_i \cdot T_{n-i} \right)$

"Form a bag of fruits with an even number of apples and an odd number of oranges"

$$S(x) = 1 + x^2 + x^4 + \ldots = \frac{1}{1-x^2}$$

$$T(x) = x^1 + x^3 + \cdots = \frac{x}{1-x^2}$$

$$S(x) \cdot T(x) = \frac{x}{(1-x^2)^2} = x + 2x^3 + 3x^5 + \ldots$$

Composition: $(S \circ P)_{n \in \mathbb{N}}$ $\quad P_0 = 0.$

"Form a vegetable garden with n plants with pots of tomatoes. Assume that pots can hold up to 3 plants."

$$S(x) = 1 + x + x^2 + \ldots = \frac{1}{1-x}$$

$\llcorner$ set of ... pots ...

$$P(x) = x + x^2 + x^3 = \frac{x(1-x^3)}{(1-x)}$$

$$S \circ P(x) = \frac{1}{1 - \frac{x(1-x^3)}{(1-x)}} = \frac{1-x}{1-2x+x^3}$$

Exponential Generating Fcts

Ground set: "Labeled" $\downarrow$ We have to consider all orderings

Product: $(S \cdot T)_{n \in \mathbb{N}} \overset{\text{def}}{=} \sum_{i=0}^{n} \frac{S_i}{i!} \cdot \frac{T_{n-i}}{(n-i)!}$

$$= \sum_{i=0}^{n} \binom{n}{i} \frac{S_i \cdot T_{n-i}}{n!}$$ $\longleftarrow$ This then goes under x^n

and shows that the prod. of 2 EGF is an EGF with "$\sum \binom{n}{i} S_i \cdot T_{n-i}$" as the seq..

"Form a password consisting of letters and numbers of length n."

S_i = seq. of numbers of length "i"

T_i = — " — letters of length "i".

$$S^{(e)}(x) = 1 + 10x + 100\frac{x^2}{2!} + \ldots = e^{10x}$$

$$T^{(e)}(x) = e^{26x}$$

$$S^{(e)}(x) \cdot T^{(e)}(x) = e^{36x}$$

Composition: "Store n distinct books on shelves of a bookshelf".

$\rightarrow$ The # shelves is either finite or we can not have empty shelves.

(This ensures the composition to be well defined