# Discrete Mathematics 1



## Solving Recurrence Relations

### C) Symbolic Differenciation Method

Let $f$ be a non-homogeneous linear recurrence with const. coefficients of order $k$;

$$f_k S_{n+k} + f_{k-1} S_{n+k-1} + \dots + f_1 S_{n+1} + f_0 = g(n).$$

$f_i \in \mathbb{R}$.

$\rightsquigarrow$ Do some algebraic manipulation to get an homogeneous recurrence.

Example1: Find a closed form for $S_n = 7S_{n-1} - 12S_{n-2} + n^2$. where $S_0 = S_1 = 1$.

1) Rewrite as $S_n - 7S_{n-1} + 12S_{n-2} = n^2$ $(*)$

2) Shift the equation by 1:

$$S_{n+1} - 7S_n + 12S_{n-1} = (n+1)^2 = \underline{\underline{n^2}} + 2n + 1$$

$\Leftrightarrow$

$$S_{n+1} - 7S_n + 12S_{n-1} - 2n - 1 = n^2. \quad (**)$$

$(*) + (**) \Rightarrow$ $S_n - 7S_{n-1} + 12S_{n-2} = S_{n+1} - 7S_n + 12S_{n-1} - 2n - 1$

$\Leftrightarrow$ $S_{n+1} - 8S_n + 19S_{n-1} - 12S_{n-2} = \underline{\underline{2n + 1}}$

(New RR).

Repeat with the new RR:

$$S_{n+2} - 8S_{n+1} + 19S_n - 12S_{n-1} = 2(n+1) + 1 = 2n+3$$

$$\Rightarrow S_{n+2} - 8S_{n+1} + 19S_n - 12S_{n-1} - 2 = S_{n+1} - 8S_n + 19S_{n-1} - 12S_{n-2}.$$

$$\Rightarrow S_{n+2} - 9S_{n+1} + 27S_n - 31S_{n-1} + 12S_{n-2} = 2$$

Again!

$h = \ldots \quad S_{n+3} - 10S_{n+2} + 36S_{n+1} - 58S_n + 43S_{n-1} - 12S_{n-2} = \underline{\underline{0}}.$

$$\chi(h) = x^5 - 10x^4 + 36x^3 - 58x^2 + 43x - 12$$
$$= (x-1)^3(x-3)(x-4)$$

By the C.P. Method; the general solution of $h$ is

$$S_n = \alpha_1 + \alpha_2 n + \alpha_3 n^2 + \alpha_4 \cdot 3^n + \alpha_5 \cdot 4^n.$$

3) We only know $S_0$ and $S_1$.

$S_2 = 7S_1 - 12S_0 + 2^2 = -1.$
$S_3 = -10.$
$S_4 = -42.$

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$S_0 = \alpha_1 + \alpha_4 + \alpha_5 = 1$
$S_1 = \alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4 + 4\alpha_5 = 1$
$S_2 = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 9\alpha_4 + 16\alpha_5 = -1$
$S_3 = \alpha_1 + 3\alpha_2 + 9\alpha_3 + 27\alpha_4 + 64\alpha_5 = -10$
$S_4 = \alpha_1 + 4\alpha_2 + 16\alpha_3 + 81\alpha_4 + 256\alpha_5 = -42$

$$\rightsquigarrow \alpha_1 = \frac{83}{54}, \quad \alpha_2 = \frac{17}{18}, \quad \alpha_3 = \frac{1}{6}, \quad \alpha_4 = \frac{-1}{2}, \quad \alpha_5 = \frac{-1}{27}.$$

$$\Rightarrow \quad S_n = \frac{83}{64} + \frac{17n}{18} + \frac{n^2}{6} - \frac{3^n}{2} - \frac{4^n}{27}.$$

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<u>Notice:</u> At every step of a "symbolic differentiation" the degree of $g(n)$ decreased by 1 <u>and</u> the order of the recurrence increased.

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<u>Example 2</u>: Find a closed form for $S_n = 9S_{n-1} - 14S_{n-2} + 5^n$ w/ $S_0 = S_1 = 1$.

1) Rewrite: $S_n - 9S_{n-1} + 14S_{n-2} = 5^n$ (*)

2) Symbolically differenciate to get:

$$S_{n+1} - 9S_n + 14S_{n-1} = 5^{n+1}$$

Multiplying by 5 (*) and equating gives:

$$h(s) = S_{n+1} - 14S_n + 59S_{n-1} - 70S_{n-2} = 0$$

$$\chi(h) = x^3 - 14x^2 + 59x - 70 = (x-2)\underline{\underline{(x-5)}}(x-7).$$

$$\Rightarrow \quad S_n = \alpha_1 2^n + \alpha_2 5^n + \alpha_3 7^n$$

3) Solving for $\alpha_1, \alpha_2, \alpha_3$ gives: $\alpha_1 = \frac{43}{15}$, $\alpha_2 = -\frac{25}{6}$, $\alpha_3 = \frac{23}{10}$

$$\Rightarrow \quad S_n = \frac{43}{15}\cdot 2^n \cancel{+} -\frac{25}{6}\cdot 5^n + \frac{23}{10}\cdot 7^n.$$

<u>Notice this.</u>

The symbolic differentiation method eliminates step-by-step the parts of "$g(n)$".

↳ It acts as a linear operator (like differentiation).

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D) Undetermined Coefficients Method

Again let $f$ be a non-hom. lin. recurrence w/ constant coefficients of order $k$:

Step 1: Find a solution $(G_n)_{n\in\mathbb{N}}$ to the homogeneous system.

Step 2: Find a "good guess" $(P_n)_{n\in\mathbb{N}}$ based on $g(n)$.

The good guesses are obtained by the symbolic differenciation of common functions.

Proposition: Suppose $(S_n)_{n\in\mathbb{N}}$ is a sequence satisfying

$$f = f_k S_{n+k} + f_{k-1} S_{n+k-1} + \dots + f_1 S_{n+1} + f_0 S_n = a^n,$$ for some $a \in \mathbb{R}$. If $a$ is a root of multiplicity $m \geq 0$ of $\chi(f_H)$, then $\chi(h)$ (the char. poly. obtained from symb. diff) has "$a$" as a root of multiplicity $m+1$.

Proof: By S.Diff.:

$$f_k S_{n+k+1} + \dots + f_1 S_{n+2} + f_0 S_{n+1} = a^{n+1}.$$

Multiplying $f$ by "$a$" and equating:

$$f_k S_{n+k+1} + \dots + f_1 S_{n+2} + f_0 S_{n+1} - a(f) = 0$$

The corresponding char. pol. is

$$f_k x^{k+1} + \dots + f_1 x^2 + f_0 x - a\chi(f)$$
$$= x(\chi(f)) - a\chi(f) = (x-a)\chi(f)$$ ⊠

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⇝ If $g(n) = a^n$, "intelligent guess" for $p(n)$ is

$$\boxed{p(n) = \alpha n^m a^n.}$$

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<u>Proposition</u> Suppose $(S_n)_{n\in\mathbb{N}}$ is a sequence satisfying
$f = f_k S_{n+k} + \dots + f_1 S_{n+1} + f_0 S_n = g_d(n)$,
where $g_d(n)$ is a polynomial in $n$ of degree $d$.
If $1$ is a root of mult. $m \geq 0$ of $\chi(f)$, then
$\chi(h)$ has $1$ as a root of mult. $m+d+1$.

the homogeneous recurrence from symb. diff.

<u>Proof</u> Induction on $d$ and use symb. diff. ⊠

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⇝ If $g(n)$ is a polynomial of degree $d$, then an "intelligent guess" for $p(n)$ would be

$$\boxed{p(n) = n^m \left(\alpha_d n^d + \alpha_{d-1} n^{d-1} + \dots + \alpha_1 n + \alpha_0\right).}$$

<u>Proposition</u> Suppose $(S_n)_{n\in\mathbb{N}}$ is a sequence satisfying

$$f = f_k S_{n+k} + f_{k-1} S_{n+k-1} + \dots + f_1 S_{n+1} + f_0 S_n = \cos\left(\frac{n\pi}{2}\right)$$

If $i$ is a root of $\chi(f)$ of mult. $m \geq 0$

then $\chi(h)$ has $i$ and $-i$ as roots of multiplicity $m+1$.

from symb. der.

<u>Proof</u> Similar to previous proofs.

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For $g(n) = \sin\left(\frac{n\pi}{2}\right)$ is similar.

The "intelligent guess" for both is

$$\boxed{P(n) = n^m\left(A i^n + B(-i)^n\right)}$$

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<u>Example 3)</u> Find a closed form for the recurrence

$$R_n = R_{n-1} + 2(n-1) \quad \forall\ n \geq 2 \quad \text{(RR)}$$

$$R_1 = 2 \quad \text{(IC)}$$

<u>Solution</u>

1) Write $R_n - R_{n-1} = 2(n-1)$

2) The char. poly of the homogeneous (RR) is

$$\chi(f_h) = x - 1$$

"1" is a root of $\chi(f_h)$ of multiplicity 1

$\Rightarrow$ The general solution to $f_h$ is

$$\boxed{R_n = \alpha_1}$$

Since $g(n) = 2n-2$ and $1$ is a root of $\chi(f_h)$,

we take

$$p(n) = \alpha_2 n + \alpha_3 n^2$$

S.P.

$\Rightarrow$ The general solution of $f$ is

$$R_n = \alpha_1 + \alpha_2 n + \alpha_3 n^2.$$

Since

$$\left.\begin{aligned} R_1 &= 2 = \alpha_1 + \alpha_2 + \alpha_3 \\ R_2 &= 4 = \alpha_1 + 2\alpha_2 + 4\alpha_3 \\ R_3 &= 8 = \alpha_1 + 3\alpha_2 + 9\alpha_3 \end{aligned}\right\| \rightarrow \begin{aligned} \alpha_1 &= 2 \\ \alpha_2 &= -1 \\ \alpha_3 &= 1 \end{aligned}$$

$$R_n = 2 - n + n^2.$$

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<u>Thm</u>: Suppose $(S_n)_{n\in\mathbb{N}}$ is a seq. satifying

$$f = f_k S_{n+k} + f_{k-1} S_{n+k-1} + \cdots + f_1 S_{n+1} + f_0 S_n = g_1(n) + g_2(n)$$

If $p_1(n)$ & $p_2(n)$ are good guesses for $g_1(n)$ & $g_2(n)$ then $p_1(n) + p_2(n)$ is ——"—— for the (RR) $f$.

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<u>Ex 4</u>: Find a closed form for the recurrence

$$S_n = 2S_{n-1} - S_{n-2} + 2S_{n-3} + 3^n + 2 \quad \text{w/ } R_0 = R_1 = R_2 = 1$$

$$\chi(f_h) = x^3 - 2x^2 + x - 2 = (x-2)(x-i)(x+i)$$

The <u>general solution</u> to $f_h$ is $\alpha_1 2^n + \alpha_2 i^n + \alpha_3 (-i)^n$.

The intelligent guess for $3^n + 2$ is: $\alpha_4 3^n + \alpha_5$.

$$S_n = \alpha_1 2^n + \alpha_2 i^n + \alpha_3 (-i)^n + \alpha_4 3^n + \alpha_5.$$

$B_3 = 32$

From:
$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1$$
$$2\alpha_1 + i\alpha_2 - i\alpha_3 + 3\alpha_4 + \alpha_5 = 1$$
$$4\alpha_1 - \alpha_2 + \alpha_3 + 9\alpha_4 + \alpha_5 = 1$$
$$8\alpha_1 - i\alpha_2 + i\alpha_3 + 27\alpha_4 + \alpha_5 = 32$$

$\Rightarrow \alpha_2 = \alpha_3$

$\Rightarrow \alpha_1 = \frac{-31}{2}, \; \alpha_4 = \frac{31}{6}, \; \alpha_5 = \frac{33}{6}$

$\Rightarrow \alpha_2 = \alpha_3 = \frac{-31}{12}$

$$\Rightarrow S_n = \left(\frac{-31}{2}\right)2^n + \left(\frac{-31}{12}\right)i^n + \left(\frac{-31}{12}\right)(-i)^n + \left(\frac{31}{6}\right)3^n + \frac{33}{6}.$$

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**Relation with ODE's:**

Ordinary Differential Equations are equations relating a function of 1 variable "$f(x)$" and its derivatives "$f'(x)$", "$f''(x)$", ...

**Fact:** When the equation is linear, the equation's solutions can be added & multiplied.

Other ODEs are more complicated and that's why ODEs and PDE's form a vast field of study.

**Example:** $\boxed{\text{ODE:}}$ $a f''(x) + b f'(x) + c f(x) = r(x)$ [Second order ODE, linear non-hom. w/ const. coeff.]

Boundary cond: [w/ $f(0) = d$]

$$f(x) = f_h(x) + f_p(x)$$

soln to hom

method of var. of parameters or inspection.

# Types of Series:

## Arithmetic Sequences:

$$A_n = A_0 + nd \quad (n \geq 0)$$

$d$: difference between two successive terms.

$A_0$: Initial Term.

You can add Arithmetic Seq. to get new ones.

If $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}}$ are A.S. then $C_n = A_n + B_n$ is an A.S

**Lemma**: If $(A_n)_{n\in\mathbb{N}}$ is an A.S. then

$$\prod_{i=0}^{n} A_i = d^{n+1} \prod_{i=0}^{n} \left(\frac{A_0}{d} + i\right)$$

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## Geometric Sequences:

$$A_n = A_0 \cdot r^n \quad (n \geq 0)$$

$r$: ratio between two successive terms

$A_0$: Initial Term

You can multiply Geometric Seq. to get new ones.

If $(A_n)_{n\in\mathbb{N}}$ & $(B_n)_{n\in\mathbb{N}}$ are G.S. then $C_n = A_n \cdot B_n$ is an G.S.

**Lemma**: If $(A_n)_{n\in\mathbb{N}}$ is a G.S. then

$$\sum_{i=0}^{n} A_i = A_0 \left(\frac{1-r^{n+1}}{1-r}\right).$$

## Linear Recurrence Relations vs Linear Algebra

Let $f$ be an homogeneous linear rec. relation w/ const. coeff. of order $k$.

$$f = f_k S_{n+k} + \cdots + f_1 S_{n+1} + f_0 S_n = 0,$$

with initial condition $(S_0, S_1, \ldots, S_{k-1})$.

We can rewrite $f$ using a matrix:

$$\underbrace{\left(\begin{array}{cccc|c} -\frac{f_{k-1}}{f_k} & -\frac{f_{k-2}}{f_k} & \cdots & -\frac{f_1}{f_k} & -\frac{f_0}{f_k} \\ \hline & & & & 0 \\ & & \mathrm{Id} & & \vdots \\ & & & & 0 \end{array}\right)}_{= M_f} \underbrace{\begin{pmatrix} S_{k-1} \\ S_{k-2} \\ \vdots \\ S_1 \\ S_0 \end{pmatrix}}_{= X} = \begin{pmatrix} S_k \\ S_{k-1} \\ \vdots \\ S_2 \\ S_1 \end{pmatrix}$$

What is the last entry of $M_f^n \cdot X$? $S_n$.

How to compute $M_f^n \cdot X$?

- $\rightsquigarrow$ Decompose into eigenspaces (diagonalization)
- $\rightsquigarrow$ All entries grow following the same rate.
- $\rightsquigarrow$ The general solution to $f$ is the expression of $X$ into the eigenbasis.

[Enlightening linear algebra excercise!]

As we have seen Recurrence Relations give rise to generalizations of A.S. and G.S..

We introduce yet another tool to solve (RR), but that will give much more.

## Generating Functions

Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers.

Def: The generating function $\mathcal{S}(x)$ of $(S_n)_{n\in\mathbb{N}}$ is the formal power series

$$\mathcal{S}(x) = \sum_{i=0}^{\infty} S_i x^i .$$

"Formal": We don't care if $\mathcal{S}(x)$ is a well-defined number; only care about the coefficients of $\mathcal{S}(x)$.

Formal Power Series Operations:

Def: Let $\mathbb{C}[[x]]$ denote the vector space of formal power series:

- $(0)_{n\in\mathbb{N}}$ is the zero-element.
- $(A_n)_{n\in\mathbb{N}}$ & $(B_n)_{n\in\mathbb{N}}$, $c\in\mathbb{C}$,

$$\sum_{i=0}^{\infty} A_i x^i + \sum_{i=0}^{\infty} B_i x^i = \sum_{i=0}^{\infty} (A_i + B_i) x^i$$

$$c \sum_{i=0}^{\infty} A_i x^i = \sum_{i=0}^{\infty} c A_i x^i .$$

**Def:** Given $\mathcal{A}(x)$ and $\mathcal{B}(x) \in \mathbb{C}[\![x]\!]$, the <u>convolution product</u> of $\mathcal{A}(x)$ and $\mathcal{B}(x)$ is (12)

$$\mathcal{A}(x)\cdot\mathcal{B}(x) := \sum_{i\geq 0}^{\infty}\left(\sum_{k\geq 0}^{i} A_k\cdot B_{i-k}\right)x^i .$$

- This is associative & distributes with $\oplus$.
- $\Rightarrow \mathbb{C}[\![x]\!]$ is an integral domain, since $\mathcal{A}(x)\cdot\mathcal{B}(x)=0 \Downarrow \mathcal{A}(x)=0$ or $\mathcal{B}(x)=0$.
- The identity is $\mathbb{1}(x) = 1$.

When is $\mathcal{A}(x)$ invertible? That is when is there $\mathcal{B}(x)\in\mathbb{C}[\![x]\!]$ such that $\mathcal{A}(x)\cdot\mathcal{B}(x) = 1$.

<u>Lemma:</u> $\mathcal{A}(x)$ has an inverse $\Leftrightarrow A_0 \neq 0$.

<u>pf</u> If $\mathcal{A}(x)\cdot\mathcal{B}(x)=1$ then $A_0\cdot B_0 = 1 \Rightarrow A_0 \neq 0$.

$\Leftarrow$ Determine $B_i$ $\forall i \geq 0$.

1) $B_0 = A_0^{-1}$.

Induction: Solve $0 = \sum_{k=0}^{i} A_k B_{i-k} \rightsquigarrow B_i$ is uniquely determine

**Def:** Given $\mathcal{A}(x)$ and $\mathcal{B}(x)\in\mathbb{C}[\![x]\!]$, the <u>composition</u> of $\mathcal{A}$ and $\mathcal{B}$ is $\square$

$$\mathcal{A}(\mathcal{B}(z)) = \sum_{i=0}^{\infty} A_i\left(\mathcal{B}(x)\right)^i .$$

The composition is well-defined when $\mathcal{A}(x)$ is a polynomial or $\mathcal{B}(0) = 0$.
(so that coefficients are finite sums).

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<u>Def</u>: Given $\mathcal{A}(x) \in \mathbb{C}[[x]]$, the <u>derivative</u> of $\mathcal{A}(x)$ is

$$\mathcal{A}'(x) = \sum_{i=1}^{\infty} i \cdot A_i \, x^{i-1} = \sum_{i=0}^{\infty} (i+1) A_{i+1} x^i .$$

The usual properties of derivative hold:

$$(\mathcal{A} + \mathcal{B})' = \mathcal{A}' + \mathcal{B}' \quad , (\mathcal{A} \cdot \mathcal{B})' = \mathcal{A}'\mathcal{B} + \mathcal{A}\mathcal{B}'$$

$$\mathcal{A}(\mathcal{B}(x))' = \mathcal{A}'(\mathcal{B}(x)) \cdot \mathcal{B}'(x).$$

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<u>Fact</u>: Two sequences are equal if and only if their generating functions are equal.

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<u>Basic Generating Functions</u>

- The <u>"empty GF"</u> $\mathbb{1}(x) := 1$ , $(A_n)_{n \in \mathbb{N}}$ $\quad A_0 = 1$, $A_i = 0 \;\; \forall i \geq 1$.

  "How many empty sets on "$i$" elements?".

- The <u>"set GF"</u> $\mathcal{E}(x) := \sum_{i=0}^{\infty} x^i$, $\quad (1)_{n \in \mathbb{N}}$

  "How many sets on "$i$" elements?"

- Take the "<u>singleton GF</u>" $\mathcal{S}(x) := x$,
  "How many sets of 1 element is there on "$i$" elements?"

$$\mathcal{S}(x) \cdot \mathcal{E}(x) = \sum_{i=0}^{\infty} \Big( \sum_{k=0}^{i} S_k \cdot \underbrace{E_{i-k}}_{=1} \Big) x^i$$

$$= \sum_{i=1}^{\infty} x^i = \mathcal{E}(x) - \mathbb{1}(x)$$

$$\Leftrightarrow -\mathcal{S}(x) \cdot \mathcal{E}(x) + \mathcal{E}(x) = \mathbb{1}(x)$$

$$\Leftrightarrow \mathcal{E}(x) = \frac{\mathbb{1}(x)}{\mathbb{1}(x) - \mathcal{S}(x)} = \frac{1}{1-x}$$

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- The "<u>subsets of an n-set</u> GF" $\mathcal{B}_n(x)$.

  Q : How many subsets of an n-set is there?
  A : The binomial coefficients.

  To form a subset, <u>for each</u> element in the n-set, it is there <u>or</u> not: that is

$$\mathcal{B}_n(x) = \big(\mathbb{1}(x) + \mathcal{S}(x)\big)^n$$

Each element is either

Out : $\mathbb{1}(x) \to$ empty

In : $\mathcal{S}(x) \to$ one singleton

by the A.P. $\mathbb{1}(x) + \mathcal{S}(x)$ represent the presence or not of an element.

By the Multiplication Principle, repeat $n$ time.

Hence $\mathcal{B}_n(x) = (1+x)^n$.

<u>Thm</u> (Division of Formal Power Series)

Let $\mathcal{A}(x), \mathcal{B}(x) \in \mathbb{C}[[x]]$ s.t. $\mathcal{A}(x) = x^i \tilde{\mathcal{A}}(x)$

$\mathcal{B}(x) = x^j \tilde{\mathcal{B}}(x)$.

and $A_i \neq 0 \neq B_j$.

The equation $\mathcal{A}(x) \cdot \mathcal{Z}(x) = \mathcal{B}(x)$ has a solution

$\Longleftrightarrow \quad i \le j$.

When $\mathcal{Z}(x)$ exists it is unique and $\mathcal{Z}(x) = \dfrac{\mathcal{B}(x)}{\mathcal{A}(x)}$

$$= x^{j-i} \frac{\tilde{\mathcal{B}}(x)}{\tilde{\mathcal{A}}(x)}.$$

<u>pf</u> <u>If $i \le j$,</u> $\mathcal{F}(x) = x^{j-i}$ is a G.F.

$\underbrace{\tilde{\mathcal{A}}(x) \text{ and } \tilde{\mathcal{B}}(x)}_{\text{Invertible}}$ are G.F. too.

$\Rightarrow \quad x^{j-i} \cdot \tilde{\mathcal{B}}(x) \cdot \tilde{\mathcal{A}}^{-1}(x)$ is a G.F.

and $\quad \mathcal{A}(x) \cdot x^{j-i} \cdot \tilde{\mathcal{B}}(x) \cdot \tilde{\mathcal{A}}^{-1}(x) = x^i \tilde{\mathcal{A}}(x) \cdot x^{j-i} \cdot \tilde{\mathcal{B}}(x) \tilde{\mathcal{A}}^{-1}(x)$.

$$= x^j \cdot 1 \cdot \tilde{\mathcal{B}}(x) = x^j \tilde{\mathcal{B}}(x) = \mathcal{B}(x).$$

It is uniquely determined.

<u>Else</u> $i > j$ If $\mathcal{Z}(x)$ is a solution

then $\left[\mathcal{A}(x) \cdot \mathcal{Z}(x)\right]_j = \left[\mathcal{B}(x)\right]_j = B_j \neq 0$

$$\left[\mathcal{A}(x) \cdot \mathcal{Z}(x)\right]_j = \left[x^i \tilde{\mathcal{A}}(x) \cdot \mathcal{Z}(x)\right]_j = \left[A_i Z_0 x^i + (\cdot) x^{i+1} + \cdots\right]_j = 0$$

$\lightning$

$\boxtimes$

- The "Sets of cardinality multiple of k GF" is

$$1 + x^k + x^{2k} + \dots = \frac{1}{1-x^k} .$$

- The "Partitions GF":

Let $\mathcal{P}(x)$ be the GF for the partitions of integers.

$$\mathcal{P}(x) = \sum_{n=0}^{\infty} p(n)\, x^n$$ , where $p(n)$ is the number of partitions of $n$.

Question: What is $p(n)$? How fast does it grow?

↳ The Man Who Knew Infinity (2016)

What is the coefficient of $x^n$?

$$x^n = x^{\lambda_1} \cdots x^{\lambda_k} , \quad \lambda \vdash n$$

How to obtain all partitions using products?
First, choose how many blocks of 1's, then 2's, ...

$$(1 + x^1 + x^2 + x^3 + \dots)(1 + x^2 + x^4 + x^6 + \dots)(1 + x^3 + x^6 + \dots)$$

....

Each term in the expansion gives 1 partition of a number.
For fixed "n", there is exactly $p(n)$ ways.

$$\Rightarrow \mathcal{P}(x) = \prod_{k=1}^{\infty} \left( \frac{1}{1-x^k} \right)$$ [Do you see the M.P. and A.P at play?]