Week 4

Discrete Mathematics 1

Multinomial coefficients:

What is $(x_1 + x_2 + \dots + x_k)^n = ?$

Def: A weak composition is a composition where parts are aloud to equal 0.

Def: Let $n \in \mathbb{N}$ and λ be a weak composition of n into k parts. The number of ways to select a λ_1-subset of $[n]$, followed by a λ_2-subset of $[n]$, ..., followed by a λ_k-subset of $[n]$ is

$$\binom{n}{\lambda_1, \dots, \lambda_k} \quad \text{[Multinomial Coefficient].}$$

Take Home: "Assigning people into named teams" because order matters.

Lemma: Let $n \in \mathbb{N}$ and λ be a weak comp. of n into k parts.

$$\binom{n}{\lambda_1, \dots, \lambda_k} = \binom{n}{\lambda_1}\binom{n-\lambda_1}{\lambda_2} \cdots \binom{n-\lambda_1-\lambda_2 \dots -\lambda_{k-1}}{\lambda_k}$$

$$= \frac{n!}{\lambda_1! \cdots \lambda_k!}$$

Pf) Follows from the Multiplication Principle.

Multinomial Thm: $\forall n \in \mathbb{N}$, we have

$$(x_1 + \dots + x_k)^n = \sum_{\substack{\lambda_1 + \dots + \lambda_k = n \\ \lambda_1, \dots, \lambda_k \geq 0}} \binom{n}{\lambda_1, \dots, \lambda_k} x_1^{\lambda_1} \cdots x_k^{\lambda_k}.$$

Corollary: $\forall\, n, k \in \mathbb{N}$, we have

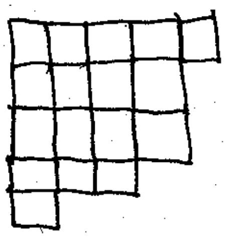
$$\sum_{\substack{\lambda_1 + \dots + \lambda_k = n \\ \lambda_1, \dots, \lambda_k \geq 0}} \binom{n}{\lambda_1, \dots, \lambda_k} = k^n.$$

Example: (Anagrams) How many anagrams are there of "Mathematics"?

Soln:
2 M's
2 A's
2 T's
1 h
1 e
1 i
1 c
1 s

$$\longmapsto \binom{11}{2,2,2,1,1,1,1,1} = 4989600.$$

Def: (Ferrers diagram) Given a partition λ of n into k parts, A Ferrers diagram is a sequence of rows containing λ_i boxes and left justified: $\lambda = 54431 \rightarrow$

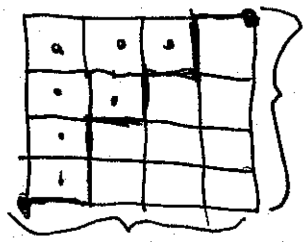


Connection to lattice paths:

$\binom{m+n}{m}$ counts the number of lattice paths from $(0,0)$ to (m,n).

The lattice paths are in bijection with partitions using at most n blocks and parts are $\leq m$.

Proof (Sketch) By picture:



$n=4$ $\lambda = 3211$

$m=4$

Recurrence Relations

Some counting problems required some equation, as opposed to a "closed form", to be solved.

Closed form: "requires finitely many steps to compute" + "no use of previous values (excepted maybe the first value.)"

Let $S_\bullet$ be a function from $\mathbb{N} \to \mathbb{N}$.

Def: A recurrence relation for S is a function that defines $S(n)$ in terms of $S(i)$'s, for $i \in [0, 1, \ldots, n-1]$.

Example: 1) Fibonacci numbers: $\underbrace{\begin{matrix} F(0)=0 \\ F(1)=1 \end{matrix}}_{\text{Initial values.}}$ and $F(n) = F(n-1) + F(n-2)$

<u>Def</u>: If R is a sequence defined recursively, i.e.

$$f(R_{n+k}, R_{n+k-1}, \dots, R_n) = g(n)$$

where f is a fct of the values in the seq. and g is a fct of n.

- If $g(n) = 0$, the recurrence is <u>homogeneous</u>.
  Else, <u>non-homogeneous</u>.
- We say that f is a <u>k^{th} order</u> recurrence relation.
- If f can be written

$$f(R_{n+k}, R_{n+k-1}, \dots, R_n) = f_k(n)\, R_{n+k} + f_{k-1}(n)\, R_{n+k-1} + \dots + f_0(n)\, R_n,$$

where the $f_i(n)$ are fcts of n that do not depend on R_n
then f is a <u>k^{th}-order linear recurrence relation.</u>
- If $f_i(n) = c_i \in \mathbb{C}$ $\forall i$ then the rec. rel. has constant coefficients.

<u>Ex</u>:
1) $F_n = F_{n-1} + F_{n-2}$ (lin. of order 2, cst coeff, homogeneous)
2) $R_n = 2R_{n-1} + 5R_{n-2} + (-1)^n$ (non-hom., 2th order lin. cst coeff)
3) $S_n = S_{n-1} + 5S_{n-2} + (-1)^n S_{n-3}$ (hom, 3^{rd} order lin,)
4) $J_n = n J_{n-1} + (-1)^n J_{n-2} + 5 J_{n-3} + 2$ (non-hom, 3^{rd} order, lin)
5) $K_n = K_{n-1} \cdot K_{n-2}$ (hom, 2^{nd} order, cst coeff)

Dérangements (bis)

Thm: The number of dérangements of $[n]$, D_n, satisfies the recurrence
$$D_n = (n-1)(D_{n-1} + D_{n-2})$$
for $n \geq 2$ with $D_0 = 1$ and $D_1 = 0$.

Pf) $\underline{n=0}$ there is only 1 dérangement: the empty dérangement.

$\underline{n=1}$: No dérangement.

It is left to check that both sides count the same set.

LHS) By def, D_n counts the dérangements of $[n]$.

RHS) Also counts this by

1) Choosing an element "k" to be placed at position "1".

↳ There are $(n-1)$ choices.

followed by 2) Arrange the remaining values in 2 disjoint ways

i) Place "1" at position "k" and the rest should be a dérangement of $[n] \setminus \{1, k\}$

↳ D_{n-2}

ii) Place "1" not at position "k".

A dérangement of $2, 3, \ldots, k-1, 1, k+1, \ldots, n$ gives the completion into a dérangement.

↳ D_{n-1}.

By the A.P. and M.P. $D_n = (n-1)(D_{n-1} + D_{n-2})$ for $n \geq 2$. $\boxtimes$

Corollary: D_n also satisfies

$$D_n = n \cdot D_{n-1} + (-1)^n \quad , \text{where} \quad D_0 = 1.$$

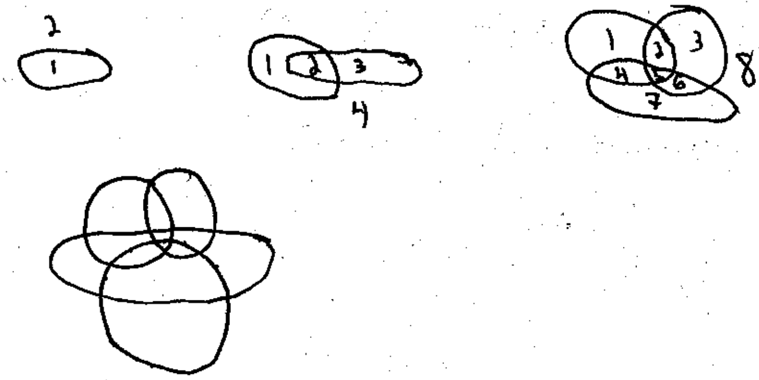
Pf] Homework.

Solving recurrence relations

A) Solving by iterations:

1) Let R_n be the # regions on the plane determined by n pairwise intersecting (in 2pts) and never 3 at the same pt.

Solution:
$R_1 = 2$
$R_2 = 4$
$R_3 = 8$
$R_4 = 14$
$R_n = ?$



The n-th ellipse has $\sim 2(n-1)$ intersection pts.
forming $\sim 2(n-1)$ arcs on the n-th ellipse.

$\longrightarrow$ Adds $2(n-1)$ regions.

Therefore
- $R_n = R_{n-1} + 2(n-1) \quad \forall\, n \geq 2$
- with initial condition $R_1 = 2$.

Now:
$$R_n = R_{n-1} + 2(n-1) = R_{n-2} + 2(n-2) + 2(n-1)$$
$$= R_{n-3} + 2(n-3) + 2(n-2) + 2(n-1).$$

$= R_1 + 2(1) + 2(2) + \cdots + 2(n-2) + 2(n-1)$

$= R_1 + 2 \cdot \frac{n(n-1)}{2} = R_1 + n(n-1)$

Since $R_1 = 2$, $R_n = 2 + n(n-1)$

- Finally check that $R_1, R_2, \ldots, R_4$ is correct.

Practical formulas:

1) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ 2) $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$

pf of 2): $\otimes$ by $1-r$ on both sides.

Ex 3) Solve (IC) $A_0 = 1$, (RR) $A_n = 3A_{n-1} + 2^n$ (for $n \geq 1$)

Soln: $A_n = 3A_{n-1} + 2^n$

$= 3(3A_{n-2} + 2^{n-1}) + 2^n$

$= 3^2 A_{n-2} + 3 \cdot 2^{n-1} + 2^n$

$\vdots$

$= 3^k A_{n-k} + 3^{k-1} \cdot 2^{n-(k-1)} + \cdots + 3 \cdot 2^{n-1} + 2^n$.

The n-th line is

$$A_n = 3^n A_0 + 3^{n-1} \cdot 2 + 3^{n-2} \cdot 2^2 + \cdots + 3 \cdot 2^{n-1} + 2^n.$$

Since $A_0 = 1$,

$$= \sum_{i=0}^{n} 2^i \cdot 3^{n-i} = 3^n \sum_{i=0}^{n} \frac{2^i}{3^i} = 3^n \left(\frac{1-(2/3)^{n+1}}{1-2/3} \right)$$

$$= 3^{n+1} \cdot \left(1 - \left(\frac{2}{3}\right)^{n+1} \right) = 3^{n+1} - 2^{n+1}.$$

<u>Def</u>: A sequence $(S_n)_{n\in\mathbb{N}}$ that satisfies a recurrence relation is a <u>particular solution</u> of it.
(it depends on initial values).

<u>Ex</u>: $a_n = 2a_{n-1}$ $(1, 2, 4, 8, 16, \ldots)$ $(3, 6, 12, 24, 48, \ldots)$ are 2 particular solutions.

The <u>general solution</u> of a recurrence relation is the expression of all particular solutions into one formula in terms of initial conditions.

In the <u>Ex 2)</u> above <u>General</u>: $R_1 + n(n-1)$
<u>Particular:</u> $2 + n(n-1)$.

<u>Theorem:</u> (Superposition Principle)
Let f be a linear recurrence relation of order k on a sequence $\{S_n\}_{n\in\mathbb{N}}$ and f_H its homogeneous version $(g(n)=0)$.

$$f(S_{n+k}, S_{n+k-1}, \ldots, S_n) = f_k(n)\cdot S_{n+k} + f_{k-1}(n)\, S_{n+k-1} + \cdots + f_0(n)\, S_n = g(n).$$

a) If $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ are particular solutions of f_H then $\forall\, \alpha, \beta \in \mathbb{R}$, $(\alpha A_n + \beta B_n)_{n\in\mathbb{N}}$ is also a part. solution of f_H.

b) If $(G_n)_{n\in\mathbb{N}}$ is the general solution of f_H and $(P_n)_{n\in\mathbb{N}}$ is a particular solution of f, then $(G_n + P_n)$ is the gen. solution of f.

Proof

a) Let $(A_n)_{n\in\mathbb{N}}$ & $(B_n)_{n\in\mathbb{N}}$ be 2 solutions for f_H:

$$f_k(n)\cdot A_{n+k} + f_{k-1}(n)\cdot A_{n+k-1} + \dots + f_0(n) A_n = 0$$
$$f_k(n)\cdot B_{n+k} + f_{k-1}(n)\cdot B_{n+k-1} + \dots + f_0(n) B_n = 0$$

Hence $\alpha A_n + \beta B_n$ satisfies

$$f_k(n)\left(\alpha\cdot A_{n+k} + \beta A_{n+k}\right) + f_{k-1}(n)\left(\alpha A_{n+k-1} + \beta B_{n+k-1}\right) + \dots + f_0(n)\left(\alpha A_n + \beta B_n\right)$$
$$= \alpha\cdot 0 + \beta\cdot 0 = 0 \qquad \forall n \geq k$$

b) As $(G_n)_{n\in\mathbb{N}}$ is the gen. soln of f_H,

$$f_k(n)\, G_{n+k} + f_{k-1}(n)\, G_{n+k-1} + \dots + f_0(n)\, G_n = 0 \qquad (\forall n \geq k)$$

and every part. solution of f_H are obtained by fixing initial conditions for G_n.

and
$$f_k(n)\, P_{n+k} + f_{k-1}(n)\, P_{n+k-1} + \dots + f_0\, P_n = g(n).$$

(as above)
$\Rightarrow$ $G_n + P_n)$ is $\underline{\underline{a}}$ solution of f.

Let Q_n be a particular solution of f.

Can we solve $\quad G_n + P_n = Q_n \quad$ for $(n=0,\dots,k-1)$?

$$\Leftrightarrow \quad G_n = \underbrace{Q_n - P_n}_{\text{Initial conditions}} \quad (n=0,\dots,k-1)$$

As G_n is the gen. sol of of f_H, this system has a solution. $\boxtimes$

B) Characteristic Polynomial Method

Let f be an homogeneous linear recurrence relation with constant coefficients of order k:

$$f(S_{n+k}, \ldots, S_n) = f_k S_{n+k} + f_{k-1} S_{n+k-1} + \cdots + f_0 S_n = 0, \qquad (S_n)_{n \in \mathbb{N}}.$$

where $f_0, f_1, \ldots, f_k \in \mathbb{R}$.

The characteristic polynomial associated to f is

$$\chi(f) := f_k x^k + f_{k-1} x^{k-1} + \cdots + f_0 .$$

The polynomial $\chi(f)$ has k roots (with multiplicities) $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$.

Thm: Let $\lambda \in \mathbb{C}$ be a zero of $\chi(f)$ of multiplicity m.

a) If $\lambda \in \mathbb{R}$, then $(\lambda^n)_{n \in \mathbb{N}}, (n\lambda^n)_{n \in \mathbb{N}}, (n^2 \cdot \lambda^n)_{n \in \mathbb{N}}, \ldots, (n^{m-1} \lambda^n)_{n \in \mathbb{N}}$ are solutions of f.

b) If $\lambda = re^{i\theta} \notin \mathbb{R}$, then $(r^n \cos n\theta), (n r^n \cos n\theta), \ldots, (n^{m-1} r^n \cos n\theta),$
$(r^n \sin n\theta), (n r^n \sin n\theta), \ldots, (n^{m-1} r^n \sin n\theta)$

are solutions of f.

c) Given k "independant" particular solutions $A_n, B_n, \ldots, K_n$ the general solution of f is

$$\Gamma_n = \alpha_1 A_n + \alpha_2 B_n + \cdots + \alpha_k K_n$$

where $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$.

Pf] (Sketch)

a)
$$f_k \cdot \lambda^n + f_{k-1}\lambda^{n-1} + \cdots + f_0 \lambda^{n-k} = \lambda^{n-k}\underbrace{\left(f_k \lambda^k + f_{k-1}\lambda^{k-1} + \cdots + f_0\right)}_{=0}$$

For $m > 1$ since $\left.\dfrac{d^{m-1}\chi(f)}{dx^{m-1}}\right|_{x=\lambda} = 0$

the same follows, for $n\lambda^n, n^2\lambda^n, \ldots, n^{m-1}\lambda^n$.

b) Then $\bar{\lambda} = re^{-i\theta}$ is also a root

- $n^j\lambda^n$ and $n^j\bar{\lambda}^n$ are solutions w/ $j = 0, 1, \ldots, m-1$
- $n^j r^n e^{in\theta}$ and $n^j r^n e^{-in\theta}$ //

Hence the sum and difference
$$n^j r^n \frac{\left(e^{in\theta} + e^{-in\theta}\right)}{2} = n^j r^n \cos n\theta$$
$$n^j r^n \frac{\left(e^{in\theta} - e^{-in\theta}\right)}{2} = n^j r^n \sin n\theta$$
are solutions for $j = 0, \ldots, m-1$.

- We have many solutions from the part a) and b)
- Get "k" particular solutions $(A_n), (B_n), \cdots (K_n)$
- Let Q_n be any solution with given $Q_0, Q_1, \ldots Q_{k-1}$.

Find a solution to
$$\begin{aligned}
\alpha_1 A_0 + \alpha_2 B_0 + \alpha_3 C_0 + \cdots + \alpha_k K_0 &= Q_0 \\
\alpha_1 A_1 + \alpha_2 B_1 + \alpha_3 C_1 + \cdots + \alpha_k K_1 &= Q_1 \\
&\vdots \\
\alpha_1 A_{k-1} + \alpha_2 B_{k-1} + \alpha_3 C_{k-1} + \cdots + \alpha_k K_{k-1} &= Q_{k-1}
\end{aligned}
\iff \det\begin{pmatrix} A_0 & \cdots & K \\ \vdots & & \vdots \\ A_{k-1} & & K_{k-1}\end{pmatrix} \neq 0$$

☒

Example 1 (Fibonacci)

$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}\ \ (\forall n \geq 2).$

Soln: $\chi(f) = x^2 - x - 1$

$f = F_{n+2} - F_{n+1} - F_n = 0$

Roots of $\chi(f)$ are $\lambda_1 = \frac{1+\sqrt{5}}{2}$

$\lambda_2 = \frac{1-\sqrt{5}}{2}$

$\Rightarrow$ Particular solutions λ_1^n and λ_2^n and the general solution of f is $\alpha\lambda_1^n + \beta\lambda_2^n$ with $\alpha, \beta \in \mathbb{R}$.

We compute α, β so that $F_0 = 0$ and $F_1 = 1$

$$\left.\begin{aligned} \alpha + \beta &= 0 = F_0 \\ \alpha\lambda_1 + \beta\lambda_2 &= 1 = F_1 \end{aligned}\right\} \text{Linear Syst}$$

$\rightsquigarrow \boxed{\beta = -\alpha} \Rightarrow \alpha\lambda_1 - \alpha\lambda_2 = 1$

$\Leftrightarrow \alpha = \frac{1}{\lambda_1 - \lambda_2} = \frac{\sqrt{5}}{5} \Rightarrow \beta = \frac{-\sqrt{5}}{5}$

$$\Rightarrow F_n = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Since $\left|\frac{1-\sqrt{5}}{2}\right| < 1$, its powers $\longrightarrow 0$.

$\Rightarrow F_n$ is the closest integer to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$.

Example 2

Find the general solution for the recurrence relation

$$A_n = 2A_{n-1} + 4A_{n-2} - 8A_{n-3} \quad (n \geq 3).$$

Solution:

Linear, homogeneous, with constant coeff. of order 3.

The characteristic polynomial of $f(A_n, A_{n-1}, A_{n-2}, A_{n-3}) =$
$A_n - 2A_{n-1} - 4A_{n-2} + 8A_{n-3} = 0$

is $\chi(f) = x^3 - 2x^2 - 4x + 8$

$= (x-2)^2(x+2) = 0$

2 is a double zero of $\chi(f)$

-2 is a simple zero. $\chi(f)$

$\Rightarrow$ 2^n, $n2^n$ and $(-2)^n$ are particular solutions

The general solution is $G_n = \alpha 2^n + \beta n 2^n + \gamma(-2)^n$, $(\alpha, \beta, \gamma \in \mathbb{R})$

Example 3) Solve the recurrence relation:

$$A_0 = 0, \quad A_1 = 1, \quad A_n = 2(A_{n-1} - A_{n-2}) \quad (n \geq 2).$$

$f(A_n, A_{n-1}, A_{n-2}) = A_n - 2A_{n-1} + 2A_{n-2} = 0$

$\rightarrow \chi(f) = x^2 - 2x + 2 = (x+1+i)(x+1-i)$

$\lambda = \sqrt{2}e^{\frac{\pi}{4}i}, \quad \overline{\lambda} = \sqrt{2}e^{-\pi/4 i}$

$\rightarrow$ We get the particular solutions

$$\sqrt{2}^n \cos\frac{n\pi}{4} \quad \text{and} \quad \sqrt{2}^n \sin\frac{n\pi}{4}$$

The general solution is

$$G_n = \alpha \sqrt{2}^n \cos \frac{\pi n}{4} + \beta \sqrt{2}^n \sin \frac{n\pi}{4}$$
$$= 2^{\frac{n}{2}} \left(\alpha \cos \frac{n\pi}{4} + \beta \sin \frac{n\pi}{4} \right)$$

We want that $A_0 = 0$ and $A_1 = 1$

$$0 = A_0 = 2^0 \left(\alpha \cos 0 + \beta \sin 0 \right) = \alpha$$
$$1 = A_1 = 2^{1/2} \left(0 \cdot \cos \frac{\pi}{4} + \beta \sin \frac{\pi}{4} \right) = \sqrt{2}\, \beta \cdot \frac{1}{\sqrt{2}} = \beta$$

The particular solution is

$$\boxed{A_n = \sqrt{2}^n \sin\left(\frac{n\pi}{4}\right)}.$$
