# Discrete Mathematics 1



## Distributions Problems: The Twelvefold Way

Let $B$ be a set of $n$ balls to be distributed into a set $K$ of $k$ boxes.

("balls" and "boxes" could be replaced by other things...)

Question: How many ways are there to distribute the "n" balls into the $k$ boxes?

To answer the question, we split it into different cases:

i) The balls are distinguishable?
ii) The boxes are distinguishable?
a) Do we forbid empty boxes?
b) Do boxes contain at most one ball?

By the M.P., there are $2 \cdot 2 \cdot 2 \cdot 2 = 16$ ways to answer the Question.

Case $*, *$, Yes, Yes: "Place exactly 1 ball in each box,
$\Rightarrow n = k$.

i) Yes, ii) Yes: This is the same as counting permutations.
(Label the balls from 1 to $n$, and the boxes as well).

$\Rightarrow n! = k!$ ways.

i) or ii) is <u>No</u>: ⇒ It is like placing red balls in ordered boxes

<u>or</u> ⇒ placing ordered balls in a selling box.

⇝ 1 way ⇒ placing red balls in a selling box.

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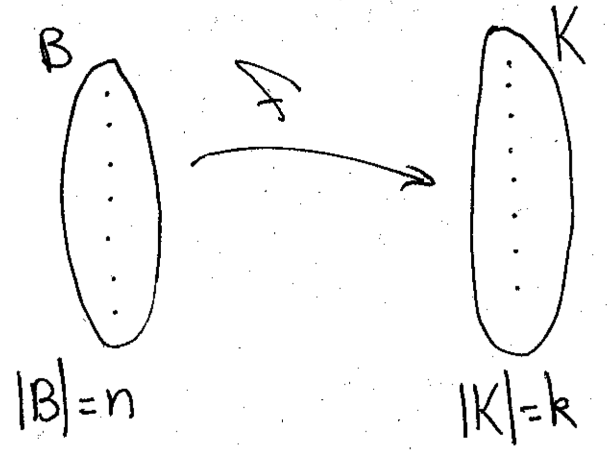
⇝ These four cases are easy/boring.

→ There are 12 remaining cases.

<u>The Twelvefold Way</u>: (by Rota, "term" from Spencer),

The distribution problem is modeled by functions:




i) Elements in B are labeled?

ii) —"— K —"—?

a) f surjective?

b) f injective?

The Case "$**$ Yes Yes" ⟺ f is bijective

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<u>Integer partition</u>: $[\lambda \vdash n]$.

Let $n \geq 1$. A <u>partition</u> $\lambda$ of $n$ is a writing

$$n = n_1 + n_2 + \dots + n_k, \text{ with } n_i \geq 1, \forall i \in [k].$$

Since the order does not matter, we assume $n_i \geq n_j, \forall i<j$.

Let $P_{n,k}$ be the number of partitions of $n$ into $k$ summands

What is $P_{n,k}$? Well... we can show a recurrence formula.

Proposition. $P_{n,k}$ satisfies

$$P_{n,k} = P_{n-k,k} + P_{n-1,k-1}$$

assuming that $P_{0,0} = 1$, $P_{n,k} = 0$, if $n \leq 0$ or $k \leq 0$.
(but not both)

Pf| Let $L_2 = \{\lambda \vdash n \mid \lambda_k = 1\}$
$L_1 = \{\lambda \vdash n \mid \lambda_i \geq 2, \forall i \in [k]\}$.

By definition we have $P_{n,k} = L_1 \sqcup L_2$.

→ From $L_1$, we can remove "1" to each part (summand) of each partition to obtain partions of $(n-k)$ still into "k" parts. That is $L_1 \approx P_{n-k,k}$.

→ From $L_2$, we can remove the last "1" from each partition to get partition of $(n-1)$ into "k-1" parts.
That is $L_2 \approx P_{n-1,k-1}$. ⊠

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A) Case "No No **":

a) Yes b) No: "Put n red balls into exactly k boxes (that are undistinguishable)".

This is exactly $P_{n,k}$. = Number of surjective maps from $[n] \to [k]$.

a) No b) Yes: "Put n red balls into k boxes, such that a box contains at most 1 ball."

If $n > k$: $= 0$.
Else: 1 (because balls and boxes are not labeled.)

Case a) No b) No: "Put n red balls into k boxes"

This can be split into k disjoint classes:
Using 1 or 2 or ... or k boxes.

By the A.P. $\longrightarrow$ $\boxed{\sum_{i=1}^{k} P_{n,i}}$

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B) Case "Yes No **"

a) Yes b) No: "Put n labeled balls into exactly k boxes, that are undistinguishable".

Set partition:

Let S be a set of cardinality n and $k \geq 1$.
A partition of S of size k is a collection of k disjoint $\underbrace{\text{subsets}}_{u_i}$ of S such that $S = \bigcup_{i=1}^{k} u_i$ and $u_i \cap u_j = \emptyset \ \forall i \neq j$

Let $S_{n,k}$ be the number of set partitions of an n-element set into k "blocks".

$\hookrightarrow$ Stirling numbers of the second kind.

What is $S_{n,k}$? $S_{0,k} = 1$, $S_{n,k} = 0$ if $n < k$

Proposition: For all n, k, we have

$$S_{n,k} = k \cdot S_{n-1,k} + S_{n-1,k-1}$$

Proof | Task: "Put n labeled balls into exactly k boxes, that are undistinguishable".

1) This is exactly $S_{n,k}$, (LHS)
2) The RHS counts the same by counting two disjoint sets:

$$Q_1 := \{ \text{partitions of } [n] \text{ into } k \text{ boxes where "1" is not alone in its block.} \}$$

$$Q_2 = \{ \text{partitions of } [n] \text{ into } k \text{ boxes where "1" is alone in its block} \}$$

- We have $|Q_2| = \boxed{S_{n-1,k-1}}$, because the other $n-1$ elements are distributed similarly into $k-1$ boxes.
- To obtain the elements in $Q_1$, we distribute the $n-1$ other elements into exactly $k$ boxes followed by picking where "1" will go.

  By the M.P. there are

  $\boxed{k \cdot S_{n-1,k}}$ ways to do this ☒

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a) No b) Yes: "Put n labeled balls into k unlabeled boxes such that each box receives at most 1 ball.

| | | |
|---|---|---|
| If $n > k$: | $= 0$ | |
| Else: | 1 | There is only one way, i.e. each ball goes into one box, (be cause boxes are unlabeled). |

Case a) No b) No : "Put $n$ labeled balls into $k$ unlabeled boxes.

This can be split into $k$ disjoint classes:

Using 1 or 2 or 3 or... or $k$ boxes.

By the A.P. $\longrightarrow$ $\boxed{\sum_{i=1}^{k} S_{n,i}}$

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Aparté: Examples of Partitions and Set-partitions

Let $n=5$. • Give all partitions of 5.

$5 = 4+1 = 3+1+1 = 3+2 = 2+2+1 = 2+1+1+1$
$= 1+1+1+1+1$

Give a formula for $\boxed{P_n = \sum_{i=1}^{n} P_{n,i}}$.

Ramanujan around 1918 gave an amazing formula with impressive precision. $\leadsto$ The Man Who Knew Infinity (2015).

• Give all set-partitions of [5].

$\{1,2,3,4,5\}$, $\{1\}\cup\{2,3,4,5\}$, $\{1,2\}\cup\{3,4,5\}$, ....
$\{2\}\cup$ ..., $\{1,3\}$
$\{3\}\cup$ ..., ⋮
$\{4\}\cup$ ..., ⋮
$\{5\}\cup$ ..., ⋮

C) Case "Yes Yes * *":

a) Yes b) No: "Put n labeled balls into exactly k labeled boxes".

This can be done in two steps

1) Case "Yes No Yes No": "Put n labeled balls into exactly k unlabeled boxes".

$\rightsquigarrow S_{n,k}$ ways.

2) Label the boxes: There are $k!$ ways to label boxes.

By the M.P. there are $\boxed{k! \cdot S_{n,k}}$ ways.

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a) No b) Yes: "Put n labeled balls into k labeled boxes such that boxes have at most 1 ball."

In "n" steps:

1) Choose the box for ball "1": $k$ ways

followed by 2) Choose the box for ball "2": $k-1$ ways

⋮

n) ——— " ——— "n": $k-n+1$ ways.

By the M.P. there are $\boxed{\frac{k!}{(k-n)!}}$ ways.

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a) No b) No: "Put n labeled balls into k labeled boxes".

$\rightsquigarrow$ Number of functions from $[n] \to [k]$.

By the M.P. $\boxed{k^n.}$ ways.

## D) Case "No Yes **":

**a) No b) No:** "Put $n$ red balls into $k$ labeled boxes"

Represent red/unlabeled balls by "o" and use separators "|"

- "$n$" times "o"
- "$k-1$" times "|"

$\Big\}$ Multiset with $n+k-1$ elements.

$$\begin{array}{cccccc} 1 & 2 & 3 & \cdots & \cdots & n+k-1 \end{array}$$

$$\underbrace{o \quad o}_{\text{Box } 1} \; | \underbrace{o \cdots o\,o\,o}_{\text{Box } 2} | | \underbrace{o \cdots o}_{\text{Box } 4} | \cdots\cdots o$$

Box 3 is empty.

We have to choose the $k-1$ places of the $n+k-1$ separators. There are $\boxed{\binom{n+k-1}{k-1}}$ ways.

Equivalently choose where the "o"'s appear: $\boxed{\binom{n+k-1}{n}}$.

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**a) No b) Yes:** "Put $n$ red balls into $k$ labeled boxes such that boxes have at most 1 ball".

$\rightarrow$ Pick $n$ of the $k$ boxes and put the balls inside those

$$\Rightarrow \boxed{\binom{k}{n}}.$$

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**a) Yes b) No:** "Put $n$ red balls into exactly $k$ labeled boxes".

Put 1 ball into each of the $k$ boxes (1 way).

Then "Put $n-k$ red balls into $k$ labeled boxes":

$$\binom{n-k+k-1}{k-1} = \boxed{\binom{n-1}{k-1}}.$$

Let B and K be finite sets and $f : B \to K$ be a function. How many functions $f$ are there? 9

## The Twelvefold Way (+4 cases)

| $\lvert B\rvert=n$, $\lvert K\rvert=k$ | f arbitrary | f injective | f surjective | f bijective |
|---|---|---|---|---|
| B labeled<br>K labeled | $k^n$<br>"Functions" | $\frac{k!}{(k-n)!}$ | $k!\,S_{n,k}$<br>"ordered set-partitions" | $n! = k!$<br>"Permutations" |
| B unlabeled<br>K labeled | $\binom{n+k-1}{k-1}$ | $\binom{k}{n}$<br>"n-subsets of k" | $\binom{n-1}{k-1}$<br>"Compositions of n into exactly k parts" | 1 |
| B labeled<br>K unlabeled | $\sum_{i=1}^{k} S_{n,i}$<br>Set partitions of [n]: Bell number | 0 or 1 | $S_{n,k} = \left\{ {n \atop k} \right\}$<br>"Set-partitions into k blocks" | 1 |
| B unlabeled<br>K unlabeled | $\sum_{i=1}^{k} P_{n,i}$<br>"Integer partitions of n" | 0 or 1 | $P_{n,k}$<br>"Integer partitions of "n" into "k" blocks" | 1 |

Examples:

1) Find the number of non-negative ($\geq 0$) solutions to

$$x_1 + x_2 + x_3 + x_4 = 20$$

Solution: - Consider the variables $x_1$ to $x_4$ as the "boxes". (labeled).
- "20" represents 20 balls to be placed, they are unlabeled.

By the T.W. there are $\binom{20+4-1}{4-1} = \binom{23}{3} = 1771$ solutions

2) Same question, but if $x_1 \geq 3$, $x_2 \geq 5$, $x_3 \geq 6$, $x_4 \geq 4$.

Solution: 1) First, we make sure that the restrictions are satisfied.
1) Place 3 balls in $x_1$
2) -"- 5 -"- $x_2$
3) -"- 6 -"- $x_3$
4) -"- 4 -"- $x_4$.

Thus we distributed 18 of the 20 balls.

2) The remaining is equivalent to "placing 2 unlabeled balls in 4 labeled urns":

$$\binom{2+4-1}{4-1} = \binom{5}{3} = 10. \quad \boxtimes$$

3) Composition**s**:

A composition is an ordered partition of n.

⇝ If the partition has k parts: ~ "Place n unlabeled balls into exactly k labeled boxes"

By the T.W. there are $\binom{n-1}{k-1}$ compositions of n into k parts.

Therefore, there are

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$$

compositions of n.

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4) What is the number of compositions of n into parts equal to 1 or 2?

<u>Solution:</u> Let $F(n)$ be that number.

$F(1) = 1$
$F(2) = 2 \rightarrow 2 = 1+1$
$F(3) = 3 \rightarrow \underbrace{3}_{X} = 2+1 = 1+2 = 1+1+1$

For n, either i) it starts with a 1
ii) it starts with a 2.

i) There are $F(n-1)$ such.
ii) —— " —— $F(n-2)$ such.

$\longrightarrow F(n) = F(n-1) + F(n-2)$.

$\rightsquigarrow$ Fibonacci numbers

5) Bell numbers

Let $n \geq 0$. $B(n)$ is the number of set-partitions of $[n]$ (Bell number).

$\rightarrow$ This is the number of equivalence relations on $[n]$.

$$B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)$$

Pf] Let "$i$" be the number of elements in $[n+1]$ not in the part containing "1".

$\rightsquigarrow$ $i \in \{0, 1, \dots, n\}$.

Let $R_i$ be the set-partitions of $[n+1]$ s.t. "$i$" elements in $[n+1]$ are not in the same part ~~than~~ as "1".

$$|R_i| = \binom{n}{i} \cdot B(i)$$

Pick the $i$ element followed by partitioning them.

By the A.P. the result follows. $\boxtimes$

## Permutations (bis)

Let $f: [n] \to [n]$ be a permutation of $[n]$

**Lemma:** $\forall i \in [n]$, $\exists k \geq 1$, s.t. $f^k(i) = i$.

Write $(i, f(i), f^2(i), \dots, f^{k-1}(i))$ (with $k$ "smallest")

This is a $k$-cycle.

Continue the process w/ $j \notin$ and until elements in $[n]$ appeared.

As a function, $f$ is a "direct-sum" of cycles (as sets and not group elements).

There are $n!$ permutations.

$\hookrightarrow$ A permutation has 1 or 2 or... or $n$ cycles.

Let $\mathcal{S}_{n,k} :=$ # permutations of $[n]$ using $k$ cycles.

"Stirling numbers of the first kind"

By the A.P.

$$n! = \sum_{k=1}^{n} \mathcal{S}_{n,k}$$

The type of a permutation is a partition of $n$, recording the length of cycles in the permutation.

$(1\,2\,3)(4\,5\,6)(7)(8\,9) \longmapsto 3321 \vdash 9$