9. Colorings of Planar graphs & Variations

Recall: $\chi(G) :=$ chromatic number of G, the smallest number of colors necessary to color the vertices of G w/o monochromatic edges.

- $1 \leq \chi(G) \leq n$.
- $1 \leq \chi(G) \leq \Delta + 1$ (by Greedy algorithm)
  $\leq \Delta$ (Brooks' Thm).
- If $K_p \subseteq G$ then $\chi(G) \geq p$.
- $\chi(G) = 2 \iff G$ is bipartite.

How many colorings does a graph G has?

Def: For $k \in \mathbb{N}$, let $P_G(k)$ denote the number of k-colorings of G.

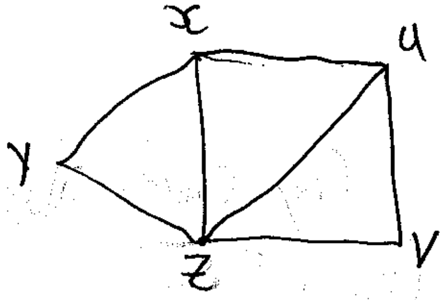
- How fast does this $P_G(k)$ grows with $k \to \infty$?
- How does $P_G(k)$ look like?

If $\chi(G) > k$, then $P_G(k) = 0$.

Ex: 1) K_n : $P_{K_n}(k) = k(k-1)\dots(k-(n-1)) = \frac{k!}{(k-n)!}$

2) N_n : $P_{N_n}(k) = k^n$ because there are no restrictions.

3)



• A) x, y and z need 3 distinct color.

$\Rightarrow k(k-1)(k-2)$

B) To color u we can choose $(k-2)$ colors (not x nor z)

c) To color v we can choose $(k-2)$ color (not u nor z's color).

$$\Rightarrow P_G(k) = k(k-1)(k-2) \times (k-2) \times (k-2) = k(k-1)(k-2)^3.$$

<u>Theorem</u> Let T be a tree of order n.
Then $P_T(k) = k(k-1)^{n-1}$.

<u>Remark:</u> It only depends on <u>n!!!</u> and not on the structure of the tree.

<u>pf</u>) Let $v_1, v_2, \dots, v_n$ be an ordering of the vertices of T s.t.
- v_1 is a leaf of T
- v_2 —"— $T \setminus \{v_1\}$
- v_3 —"— $T \setminus \{v_2, v_3\}$.

Start by giving a color to v_n : k choices.
Then $\forall\, i \in \{n-1, n-2, \dots, 2, 1\}$, add v_i to the graph containing $v_{i+1}, \dots, v_n$. Since v_i is a leaf only one color is forbidden $\Rightarrow$ $k-1$ choices

$\overset{\text{M.P.}}{\Rightarrow}$ $k(k-1)^{n-1}$.

$\square$

So far, we have seen that $P_G(k)$ is a polynomial. In fact, this is <u>always</u> true.

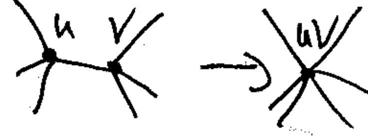
It means that $\chi(G)$ is the smallest $\oplus$ integer which is not a root of $P_G(k)$.

$\leadsto$ Birkhoff introduced it to attack the 4-color conjecture

<u>Deletion - Contraction principle</u>: $G = (V, E)$

Let $G - uv := (V, E \setminus \{u, v\})$ (Deletion)

$G / uv := ((V \setminus \{u, v\}) \cup \{uv\}, E')$ (Contraction).



Then the k-coloring of $G - uv$ are partitioned into 2 classes

$C(k) \leftarrow$ u and v have the same color

$D(k) \leftarrow$ 〃 have diff colors.

$$\Rightarrow P_{G-uv}(k) = |C(k)| + |D(k)|$$

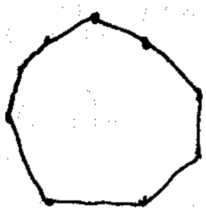
But $P_G(k) = |C(k)|$ (Clear).

also $P_{G/uv}(k) = |D(k)|$

$$\Rightarrow P_G(k) = P_{G-uv}(k) - P_{G/uv}(k).$$

Ex: 1) K_3  $P_{K_3}(k) = P_{\Lambda}(k) - P_{\vdots}(k)$

$$= k(k-1)^2 - k(k-1)^1 = k(k-1)[(k-1)-1]$$
$$= k(k-1)(k-2).$$

2) C_n  $P_{C_n}(k) = P_{\vdots}(k) - P_{C_{n-1}}(k)$

$$P_{C_n}(k) = k(k-1)^{n-1} - P_{C_{n-1}}(k)$$

$P_{C_3}(k) = P_{K_3}(k) = \boxed{k(k-1)(k-2)}$

$$P_{C_4}(k) = P_{\sqcup}(k) - P_{K_3}(k) = k(k-1)^3 - k(k-1)(k-2)$$
$$= k(k-1)[(k-1)^2-(k-2)]$$
$$= k(k-1)(k^2-3k+3)$$

$$P_{C_n}(k) + P_{C_{n-1}}(k) = k(k-1)^{n-1}$$

(RR) Nice! $\chi_R(x) = x+1 \Rightarrow \alpha(-1)^n$ is the general solution to the homogeneous RR.

$g(n) = k(k-1)^{n-1} \Rightarrow$ particular solution: $\beta \cdot k(k-1)^{n-1}$

$\underline{n=3}$: $-\alpha + \beta k(k-1)^2 = k(k-1)(k-2)$

$\underline{n=4}$ $\alpha + \beta k(k-1)^3 = k(k-1)(k^2-3k+3)$

$\Rightarrow \boxed{\alpha = \beta k(k-1)^2 - k(k-1)(k-2)}$

$$\beta k(k-1)^2 + \beta k(k-1)^3 - k(k-1)(k-2) = k(k-1)(k^2-3k+3)$$

$$\beta = \frac{k(k-1)(k^2-3k+3) + k(k-1)(k-2)}{k(k-1)^2[1 + k-1]}$$

$$= \frac{\cancel{k}(\cancel{k-1})[k^2-3k+3+k-2]}{k^{\cancel{2}}(k-1)^{\cancel{2}}} = \frac{(k^2-2k+1)}{k(k-1)} = \boxed{\frac{k-1}{k}}$$

$$\Rightarrow \alpha = \frac{k-1}{k} \cdot k(k-1)^2 - k(k-1)(k-2) = (k-1)^3 - k(k-1)(k-2)$$

$$= (k-1)[(k-1)^2 - k(k-2)] = \boxed{\cancel{k}\, k-1}$$

$$\Rightarrow (k-1)(-1)^n + \frac{k-1}{k} \cdot k(k-1)^{n-1} = \boxed{(k-1)(-1)^n + (k-1)^n}$$

<u>Proposition</u>: Let G be a graph with clique number $w(G) = r$.
The chromatic polynomial of G is divisible by

$$\frac{k!}{(k-r)!}$$

<u>Pf</u> • Locate a r-clique. it has $\frac{k!}{(k-r)!}$ colorings

• For each coloring of the clique it can be extended by $q(k)$ coloring for the rest.

$$\overset{M.P.}{\Rightarrow} \frac{k!}{(k-r)!} \cdot q(k) = P_G(k). \quad \boxtimes$$

Some properties of $P_G(k)$: Say G is connected.

- Constant coeff is zero.
- Coeff of k^1 to k^n are non-zero.
- Leading coefficient is 1.
- Coeff. of k^{n-1} is $-m$ $m = \#$ edges.
- Coeff alternate in sign.
- The abs. coefficients form a log-concave sequence. (conjectured in 1968).
  - ↳ June Huh (2010-12 J. AMS).
  - ↳ w/ Adiprasito and Katz generalized to matroids.

Log-concave: $a_0 \leq a_1 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq a_{i+2} \geq \cdots \geq a_n$

$\forall\, i \in [1, 2, \ldots, n-1]$: $a_{i-1}\, a_{i+1} \leq a_i^2$

Chromatic polynomial $\longleftrightarrow$ Char. poly of a matroid defined using Möbius function on the "flats".

Now, lets prove that planar graphs have chromatic number at most 5.

<u>Observe:</u>

<u>Lemma 1</u> Let G be a planar graph.

Then $\omega(G) \leq 4$.

<u>Pf</u> Since K_5 is not planar G can not have a clique of size 5 or more ☒

<u>Technical lemma:</u> Let $G=(V,E)$ be a graph and assume it has a k-coloring.

Pick two colors "red", "blue" and a connected component C of the induced subgraph of G on red-blue vertices.

Switching red $\Leftrightarrow$ blue on C gives a proper coloring of G.

<u>Pf</u> Assume $x-y$ is monochromatic (say red).

- If $x, y \in C \Rightarrow x, y$ were both blue $\lightning$
- If $x, y \notin C \Rightarrow x, y$ were both red $\lightning$
- If $x \in C$, $y \notin C \Rightarrow x, y$ are connected, x was blue, y was red $\Rightarrow x, y \in C$. $\lightning$

☒

Thm: The chromatic number of a planar graph is at most 5.

pf Induction on the order n.

If $n \leq 5$ then $\chi(G) \leq 5$.

By previous Thm, G has a vertex x of degree at most 5.

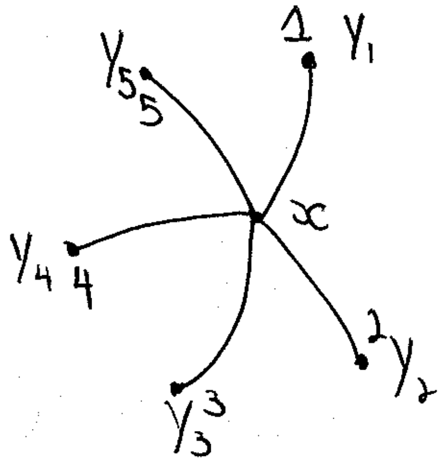
Let $H := G \setminus x$. (induced)

By induction H has a 5-coloring.

- If $\deg(x) \leq 4$, we have a free color for x. ✓
- Else $\deg(x) = 5$.
- Say y_1, y_2, y_3, y_4, y_5 are neighbors.

If less than 5 colors are used, x has a free color. ✓

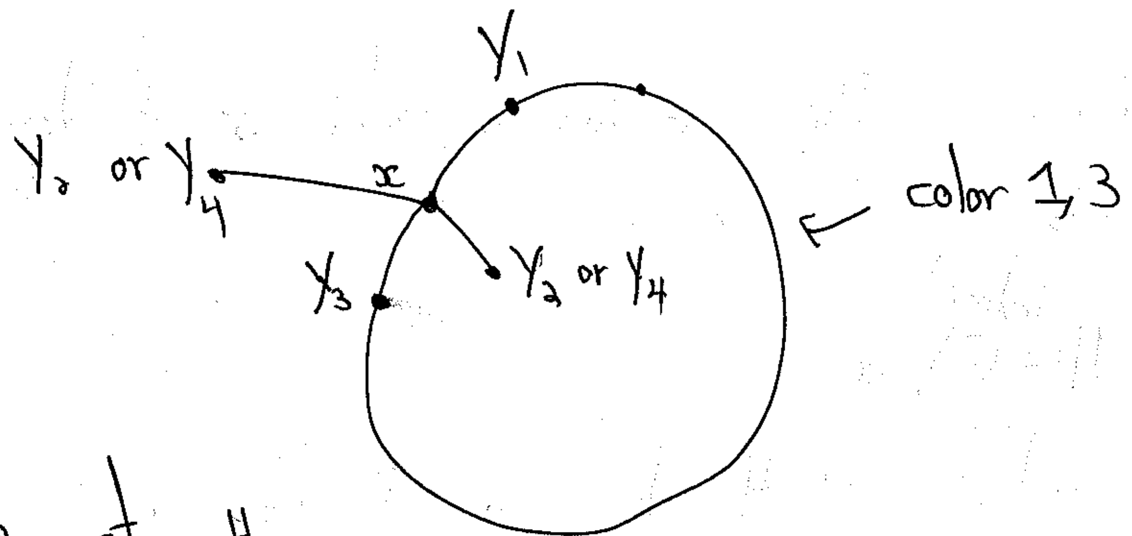
So



Consider $H_{1,3}$ the s.g. of H induced by color 1, 3.

- If Y_1 and Y_3 are not in same c.c.
  - Apply <u>Tech. lemma</u> and now have 1 free color.

Else



Now look at $H_{2,4}$

- If Y_2 and Y_4 are not in same c.c. we won. ✓ (Tech Lemma)
- Else there is a plane curve with vertices of color 2, 4 ↯

So we can give color "2" to x. ☒

Algorithm for computing the chromatic polynomial of a graph

Let $G=(V,E)$

1) Put $\mathcal{G} = \{(+,G)\}$
2) While there is a graph in $\mathcal{G}$ which is not a null graph, do:
   i) Pick $(\varepsilon, H) \in \mathcal{G}$, $H \neq N_n$. and $e \in E_H$.
   ii) Remove (ε, H) from $\mathcal{G}$ and add:
   $$(\varepsilon, H-e),\ (-\varepsilon, H/e) \text{ to } \mathcal{G}.$$
3) Put $P_G(k) = \sum \varepsilon k^p$, summing over elements (ε, H) in $\mathcal{G}$ and H has order p.