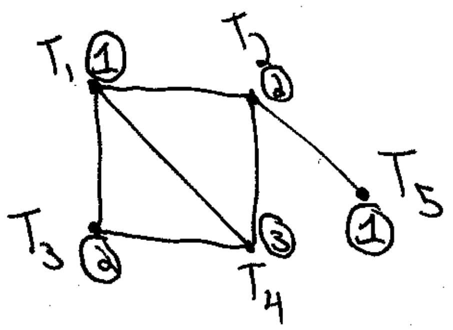
7. Colorings of graphs

Suppose we have to schedule some tasks $T_1, T_2, \dots,$ but we have a limited amount of, say power, supply for them to be executed. For example T_1 has a conflict w/ T_3, T_2 w/ T_4, etc.

In how many "period" can it be done while avoiding conflicts?



Edge: conflict.

$\Rightarrow$ 3 "periods".

Period 1: T_1 and T_5
Period 2: T_2 and T_3
Period 3: T_4.

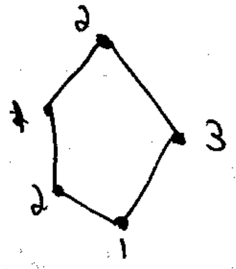
Def:
- A vertex-coloring of a simple graph G is a map $\varphi : V \to \mathbb{N}$ such that $\varphi(v_1) \neq \varphi(v_2)$ when $\{v_1, v_2\} \in E$.
- If $|\varphi(V)| = k$, φ is a k-coloring.
- If G has a k-coloring, we say it is k-colorable.
- The smallest k s.t. G is k-colorable is the chromatic number of G, noted $\chi(G)$.

Examples: • K_n : $\chi(K_n) = n \quad \forall n \geq 1$.

• Null graph. $(E = \emptyset)$. $\chi(N_n) = 1$

• C_{2n} with $n \geq 1$. $\chi(C_{2n}) = 2$.

• C_{2n+1} with $n \geq 1$. $\chi(C_{2n+1}) = 3$.



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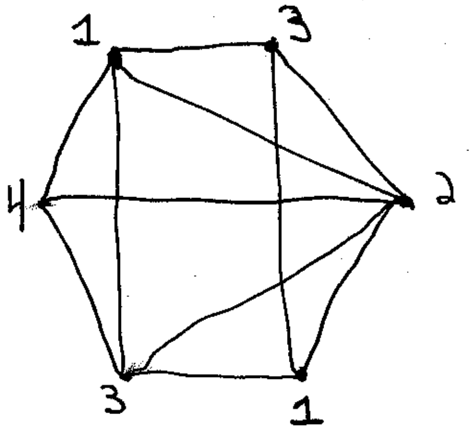


It has a K_4 subgraph.

Thm: Let G be a graph of order $n \geq 1$.
Then $1 \leq \chi(G) \leq n$.

Moreover $\chi(G) = n \iff G \cong K_n$ and $\chi(G) = 1 \iff G \cong N_n$.

Proof: • $1 \leq \chi(G)$ is obvious. (Need at least 1 image)
• $\chi(G) \leq n$: assigning diff colors to each vertex is a valid coloring.

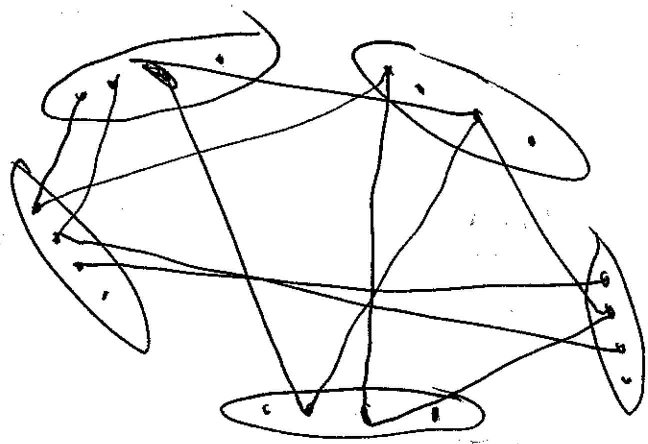
• In K_n no two vertices can receive the same color $\Rightarrow \chi(K_n) = n$

• Suppose $G \not\cong K_n$, then $\exists\, x \neq y$ and $\{x, y\} \notin E$. Assigning $\varphi(x) = \varphi(y)$ and all other to $n-2$ diff colors is a valid coloring $\Rightarrow \chi(G) \leq n-1$.

- Assigning all vertices of N_n to the same color works.
  $\Rightarrow \chi(N_n) = 1$
- Suppose $G \neq N_n$, $\Rightarrow \exists\ x \neq y$ s.t. $\{x, y\} \in G$
  $\Rightarrow$ x and y can not be the same color
  $\Rightarrow \chi(G) \geq 2$. $\boxtimes$

<u>Corollary</u>: Let G be a graph and H a subgraph of G.
Then $\chi(G) \geq \chi(H)$.
If G has a subgraph equal to a complete graph K_p.
<u>then</u> $\chi(G) \geq p$.

Given a k-coloring of G, partition $V = V_1 \sqcup V_2 \sqcup, \dots \sqcup V_k$
according to the colors: (color-partition)



The induced subgraphs G_{V_i} are null graphs.

"$\chi(G)$ is the smallest integer k s.t. V can be partitionned into k sets with each set inducing a null graph."

<u>Def</u>: The <u>clique number</u> $\omega(G)$ is the size of the largest induced complete subgraph in G.

$$\Rightarrow \quad \chi(G) \geq \omega(G).$$

<u>Corollary</u>: Let G be a graph of order n and let q be the largest order of an induced subgraph of G equal to a null graph N_q.

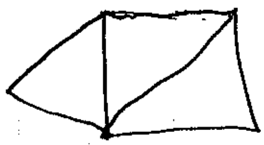
Then $\chi(G) \geq \lceil \frac{n}{q} \rceil$.

<u>Proof:</u> Let $\chi(G) = k$ and $V_1 \sqcup \cdots \sqcup V_k$ be a color partition. Then $|V_i| \leq q$. $\forall i$.

$$n = |V| = \sum_{i=1}^{k} |V_i| \leq \sum_{i=1}^{k} q = kq.$$

$\Rightarrow \frac{n}{q} \leq k = \chi(G)$. Since $\chi(G) \in \mathbb{N}$ it follows. $\boxtimes$.

<u>Ex:</u>



$q = 2 \quad \Rightarrow \chi(G) \geq \lceil \frac{5}{2} \rceil = 3.$

We know which graphs have $\chi(G)=1$ and $\chi(G)=n$.
What about $\chi(G)=2$?

Thm: Let G be a graph with at least one edge.

$$\chi(G) = 2 \iff G \text{ is bipartite}$$

Proof: How to prove $\chi(G)=k$:

1) Give a k-coloring
2) Show that no $k-1$ coloring excists.

- Assume G is bipartite.
  Color the left vertices blue and right vertices red.
  $\Rightarrow$ This is a valid 2-coloring $(\chi(G) \leq 2)$.
  $\Rightarrow$ Since it has at least one edge $G \neq N_n$ and by Thm $\chi(G) \geq 2$.
  $\Rightarrow \chi(G)=2$.
- The 2-coloring of G gives a color partition into 2 blocks $\Rightarrow$ G is bipartite. ⊠

Hence $\chi(G)=2 \iff$ Every cycle in G has even length.

How to get a coloring? (• Determine if G has a k-coloring is NP-complete
• Chromatic number is NP-hard).

Greedy algorithm for vertex-coloring

Input: G a graph and an ordering of V; $v_1, v_2, \dots, v_n$.

Output: A coloring of G

Step 1: $\varphi(1) := 1$

Step 2: $\forall\, i \geq 2$, • let p be the smallest color such that none of the neighbors of v_i in $v_1, \dots, v_{i-1}$ is colored p ("the first available color").
set $\varphi(v_i) := p$.

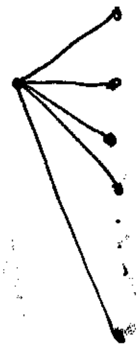
Thm: Let G be a graph s.t. max. degree of a vertex is Δ. Then the greedy algorithm gives a $(\Delta+1)$-coloring (or smaller) and so $\chi(G) \leq \Delta + 1$.

Pf: • It always assigns a diff. color on edges hence it returns a coloring.
• There are at most Δ vertices adjacent to v_i so at most $\Delta+1$ colors are necessary at each step. Further there is always one available to be choosen ⊠

The result of greedy highly depend on the order.

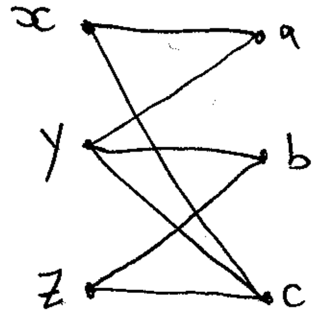
Say that G is k-colorable and $V = V_1 \sqcup V_2 \sqcup \dots \sqcup V_k$ is the color partition. If the order is: all vertices of V_1 then $V_2 \dots$, until V_k. Then it will give a k coloring.

Ex: $\cdot K_{1,n}$



$\Delta = n$ greedy still gives always a 2-coloring.

$\cdot$



Order: x, a, b, y, z, c

Colors assigned : $1, 2, 1, 3, 2, 4$

Thm: Let G be a graph s.t. max. degree of a vertex is Δ. If G is connected and not regular, then $\chi(G) \leq \Delta$.

Pf] Since it is not regular, there is a vertex v_n s.t. $\deg(v_n) \leq \Delta - 1$.

- List all its neighbors $v_{n-1}, v_{n-2}, \dots,$
- There are at most $\Delta - 1$ of them.
- Next list (backwards) the neighbors of v_{n-1}, and continue with v_{n-2}, etc.
- Since G is connected all vertices will eventually be listed
- Every vertex v_i has at most $\Delta - 1$ neighbors appearing before it in v_1 to v_{i-1} by construction.

$\Rightarrow$ Doing the greedy algorithm will give a Δ-coloring. ⊠

8. Planar & Plane graphs

$\rightarrow$ We consider only simple graphs (no loops or multi-edges).

Consider the plane $\mathbb{R}^2$.

Under which conditions is it possible to "draw" a graph on $\mathbb{R}^2$?

$\rightarrow$ Say you want to design a circuit board, circuit are not allowed to intersect.

<u>Def:</u> Let $G=(V,E)$. A <u>drawing</u> of G is a function (injective) $p: V \to \mathbb{R}^2$ along with continuous injective functions

$$q_e : [0,1] \to \mathbb{R}^2 \quad \forall e \in E \qquad \text{(called } \underline{\text{arcs}}\text{)}$$

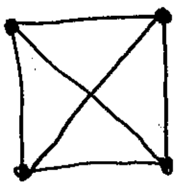
$$e = \{a,b\} \quad \text{such that} \quad \begin{aligned} q_e(0) &= p(a) \\ q_e(1) &= p(b). \end{aligned}$$

(and no arc contains vertices).

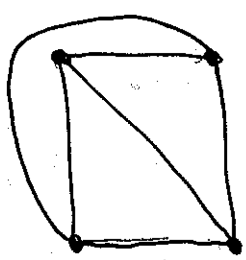
G + drawing =: "<u>Topological Graph</u>"

- A drawing is <u>planar</u> when the only common points between two arcs are the endpoints or they don't intersect.
- A graph is <u>planar</u> if it has at least one planar drawing.
- A <u>plane graph</u> is a graph with a specific planar drawing.

<u>Ex:</u>

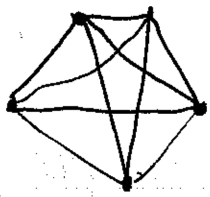


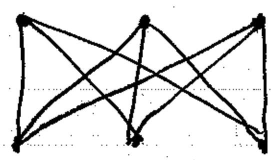
K_4



K_4

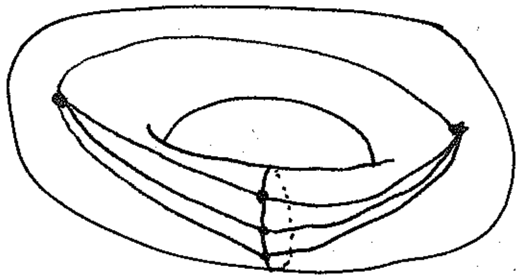
K_5 ?

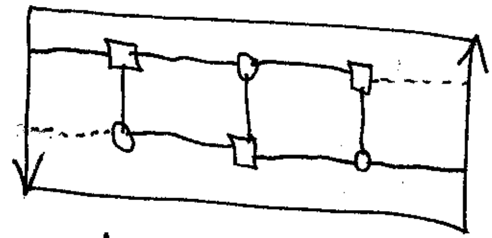




$K_{3,3}$

We can draw K_5 and $K_{3,3}$ on different surfaces:



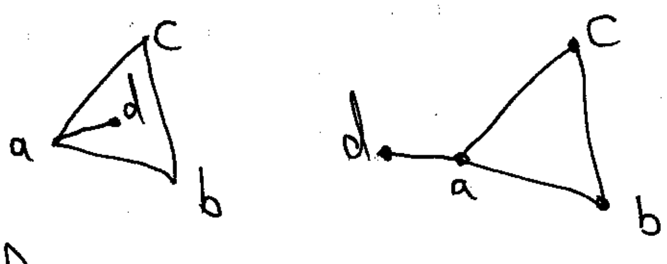


Möbius Strip

<u>Def</u>: The connected regions of $\mathbb{R}^2 \setminus G$ (plane graph) are called <u>faces</u> of the drawing of G.

There are bounded and $\underbrace{\text{unbounded}}_{\substack{\text{(only 1)} \\ \text{by compactness}}}$ regions.

<u>Warning</u>: In general faces depend on the drawing:



For a face, we can write the boundary as a cycle of the graph.

Let $f_1, f_2, \ldots, f_r$ be the number of edge-curves for each face of a plane graph or order n w/ e edges.

Then $f_1 + f_2 + \dots + f_r = 2e$.

Thm: Let G be a plane graph of order n with e edges curves and assume G is connected.

Then the # of faces "r" of G is

$$r = e - n + 2.$$

Pf)

- Assume G is a tree.

  Then $e = n-1$

  $r = 1 \leftarrow$ There is only 1 region. ✓

- Assume G is not a tree.

  It has a spanning tree T with $n' := n$

  $e' := n-1$ edges

  $r' := 1$ region.

  and $r' = e' - n' + 2$.

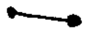
Now add each edge-curve missing. Each time e' increase by one and the number of regions too. and n' stays the same.

Hence $r' = e' - n' + 2$ remains true.

⊠

<u>Thm</u>: Let G be a connected plane graph.
G has a vertex of degree at most 5.

<u>Pf</u> · Since G has no loops no region has only 1 edge-curve.

· Since G has no multiedge no region has 2 edge-curves.
(except ) which is fine

· In $f_1 + f_2 + f_3 + \cdots + f_r = 2e$

$$f_i \geq 3.$$

$$\Rightarrow \quad 3r \leq 2e \quad \Leftrightarrow \quad r \leq \frac{2e}{3}.$$

<u>Euler</u>: $\boxed{r = e - n + 2}$

$$\frac{2e}{3} \geq e - n + 2 \quad \Leftrightarrow \quad n \geq \frac{e}{3} + 2$$

$$\Leftrightarrow \quad 3n \geq e + 6$$

$$\Leftrightarrow \quad e \leq 3n - 6.$$

By handshake lemma

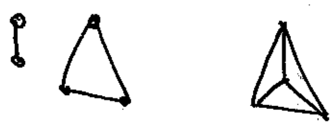
$$\sum_{v \in V} \deg(v) = 2e$$

$$\frac{\sum \deg(v)}{n} = \frac{2e}{n} \leq \frac{6n-12}{n} = 6 - \frac{12}{n} < 6.$$

Since the average degree is less than 6 at least one vertex has to have degree 5 or less $\boxtimes$.

<u>Example</u>:

- K_n is planar if and only if $n \leq 4$.

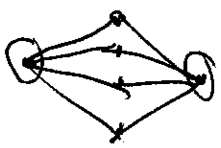
$\Leftarrow$ 

$\Rightarrow$ By Euler $e \leq 3n - 6$

$10 \leq 3 \cdot 5 - 6 = 9$ ↯

Since K_5 is not planar any K_n, $n \geq 5$ won't be.

- $K_{p,q}$ is planar $\Leftrightarrow$ $p \leq 2$ or $q \leq 2$.

$\Leftarrow$ Easy 

$\Rightarrow$ In each bipartite graph, there are no odd cycles
$\Rightarrow$ # edge-curves is at least 4 for each region.

$\Rightarrow$ $r \leq \frac{e}{2}$

Euler: $\frac{e}{2} \geq e - n + 2 \iff 2n - 4 \geq e.$

Since $K_{3,3}$ has $n = 6$, $e = 9$ $K_{3,3}$ is not planar

$\Rightarrow K_{p,q}$ $p \geq 3$ and $q \geq 3$ is not planar. $\square$

A subdivision of a graph:

Thm: (Kuratowski, 1930) (in French)

A graph G is planar $\iff$ it does not have a subgraph which is a subdivision of a K_5 or of a $K_{3,3}$.
