## 5. Bipartite graphs & Matching problems

**Recall:** Relation vs bipartite graphs:

- Given two (finite) sets $X, Y$ a relation is a subset $R \subseteq X \times Y$.
- This relation give rise to a bipartite graph $G_R = (X \cup Y, E)$ on the vertex set $X \cup Y$ where $E = \{ \{x,y\} \in X \times Y \mid (x,y) \in R \}$.

It is possible that $X = Y$, but we focus here on the case $X \neq Y$

- And vice-versa, connected bipartite graphs define a relation (or incidence structure).

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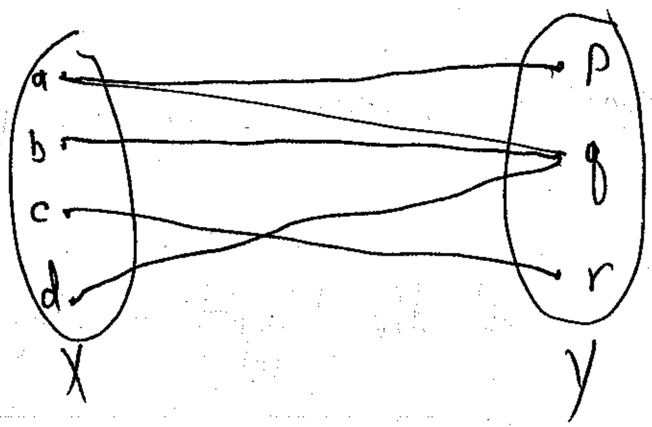
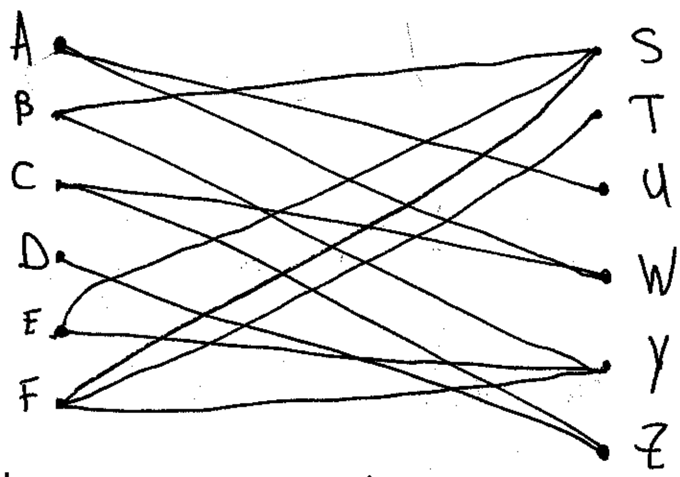
Examples: 1)



2) Alice, Bianca, Chloe, David, Elliot and Frederik want to get married after June $30^{th}$ 2017. They all have their personal preferences among Uli, Theresa, Stefany, Willam, Yulia, Zack.




Is it possible to completely match A to F to a partner on the right, while respecting their preferences?

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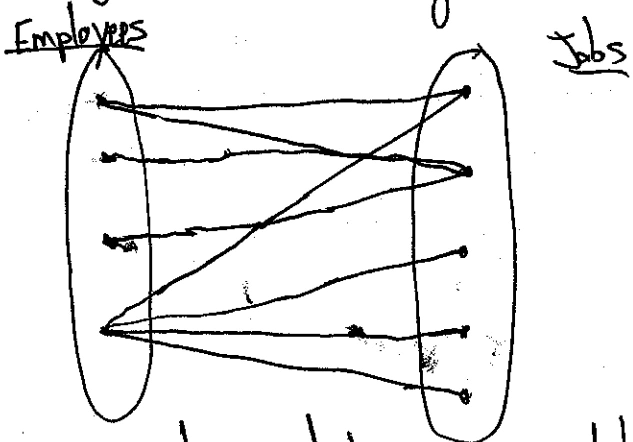
3) Suppose you have 4 employees each able to do some tasks among 5 different "jobs".




What is the maximal matching possible (assign the most employees to jobs)?

Or: Given a relation, find the "biggest" function in it.

**Lemma:** Given a bipartite graph $G=(X \cup Y, E)$, then

$$\sum_{x \in X} \deg(x) = \sum_{y \in Y} \deg(y) = |E|.$$

**Pf]** Double counting on edges.

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**Ex:** Say that each employee can do $k$ jobs and each job can be done by $k$ employee, then

a) # employee = # jobs.

b) for any $n$-subset of employees, there is always at least $n$ jobs for which some member is qualified.

a) By lemma: $|X| \cdot k = |Y| \cdot k = |E|$

b) Let $A \subseteq X$ and $J(A) = \{ y \in Y \mid \{x,y\} \in E \text{ for some } x \in A \}$.

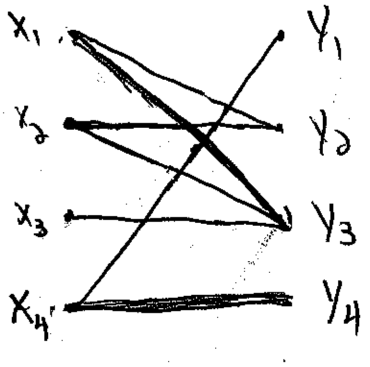
- The number of edges ending in $A$ is $k \cdot |A| = kn$.
- Each of these have one vertex in $J(A)$.
- The same condition on $J(A)$ says that there are $k \cdot J(A)$ edges w/ one vertex in $J(A)$.

$$\Rightarrow |E_A| = kn \le k|J(A)| \Leftrightarrow n \le |J(A)|.$$

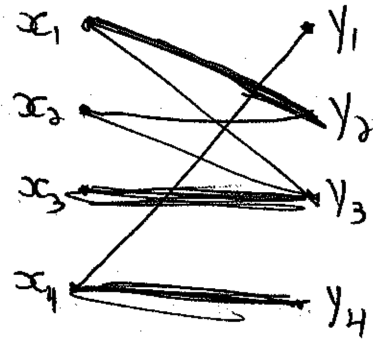
**Def:** A <u>matching</u> in a bipartite graph $G = (X \cup Y, E)$ is a subset $M \subseteq E$ with the property that no two edges in $M$ have a common vertex.

**Ex:**

$M_1$:




$M_2$:

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**Def:**
- A matching $M$ is a <u>maximum matching</u> for $G$ if no other matching of $G$ has greater cardinality.
- A matching is <u>complete</u> if $|M| = |X|$. (every "employee" gets assigned a job).

For $M_2$, $J(\{x_1, x_2, x_3\}) = \{y_2, y_3\}$ 3 employees can do 2 jobs

If $|J(A)| < |A|$ someone will be disappointed.

<u>Hall's condition:</u> If $G$ has a complete matching,
(1935) then $|J(A)| \geq |A|$, $\forall A \subseteq X$.

# Hall's condition is sufficient!

<u>Hall's Theorem</u>: The bipartite graph $G = (X \cup Y, E)$ has a complete matching $\Longleftrightarrow$ $|J(A)| \geq |A|$, $\forall A \subseteq X$.

<u>Proof</u>: $\Rightarrow$] Given the matching, $\forall A \subseteq X$, the vertices in $Y$ matched to $A$ form a subset of $J(A)$ of size $|A|$.

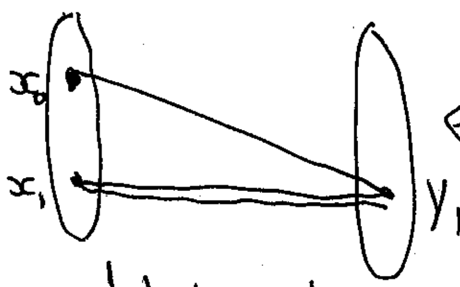
$\Leftarrow$] <u>Idea</u>: given a matching $M$ such that $|M| < |X|$, construct a matching $M'$ such that $|M'| = |M| + 1$.

<u>Step 1</u>:
- Pick $x_0 \in X$, not matched by $M$.
- Since $|J(\{x_0\})| \geq |\{x_0\}| = 1$, there is an edge $(x_0, y_1) \in E$.
- If $y_1$ is unmatched, add $\{x_0, y_1\}$ to $M$ and we are done.

<u>Step 2</u>: If $y_1$ is matched, say to $x_1$, then

$$|J(\{x_0, x_1\})| \geq |\{x_0, x_1\}| = 2$$

has to have $y_2$ connected to either $x_0$ or $x_1$.



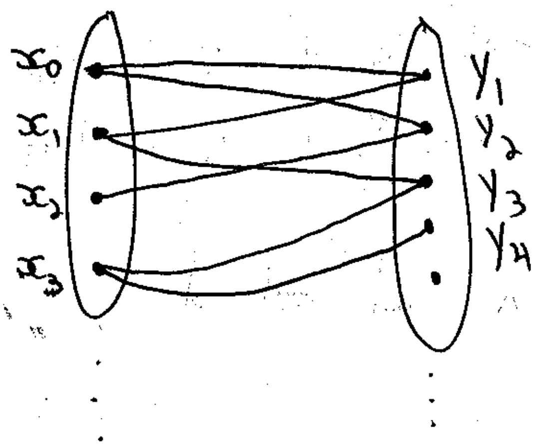

If $y_2$ is unmatched, match it to the neighbor and adapt $y_1$.

<u>Step 3</u>: If $y_2$ is matched, iterate. Since $G$ is finite there has to have an unmatched vertex $y_r$ (using $|J(A)| \geq |A|$)

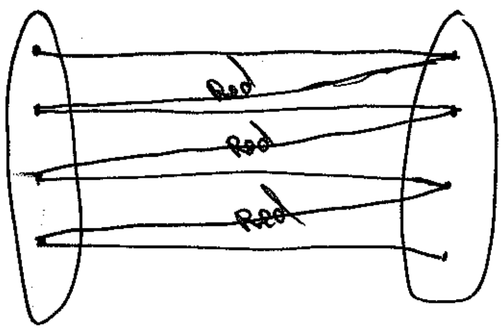
<u>Step 4</u>: Each vertex $y_i$ $(0 \leq i \leq r)$ is adjacent to at least one $x_0, x_1, \ldots, x_{i-1}$.

$$\leadsto \overbrace{y_r\ x_s}^{\notin M}\ \overbrace{y_s\ x_t}^{\notin M}\ y_t \cdots, \overbrace{y_w, x_0}^{\notin M}$$

$$\underbrace{x_s\ y_s}_{\in M} \qquad \underbrace{x_t\ y_t}_{\in M}$$




We have an alternating path from "$\notin M$", "$\in M$" that starts with and end with "$\notin M$":




Replace the "red" edges by the black one to increase the matching.

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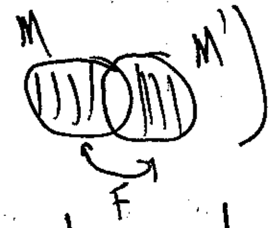
Call the paths in the proof <u>alternating path for $M$</u>.

↳ The theorem shows that if Hall's condition is satisfied <u>and</u> you have an incomplete matching, you can find an alternating path for $M$.

<u>Thm</u>: If $M$ is a non-maximum matching in a bipartite graph $G$, then $G$ contains an alternating path for $M$.

<u>Proof</u>:

Let $M^*$ be a maximum matching.

Define $F = M \Delta M^*$ (symmetric difference: 

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- The edges and vertices of $F$ form a graph where vertices have degree 1 or 2. $\Rightarrow$ Components are chains or cycles.
- In each path or cycle, the edges alternate between $M$ and $M^*$.
    - In cycles, their number is equal (Why?)

Since $|M^*| > |M|$ there is at least one component which is a path and alternating path for $M$. ⊠.

## Algorithm to get a maximum matching:

**Step 1:** Start with one edge $e \in E$.

**Step 2:** · Search for alternating path
· If found replace current matching and repeat
· Else stop and return current matching.

### Search for alternating path: (Breadth first search).

1) Take $x_0$ unmatched.
2) At level 1 put all adjacent vertices $y_1, \dots, y_k$ of $x_0$
if one is unmatched, stop: $x y_i$ is alternating.
3) If all are matched at level "$i$" insert all their matched "$x$'s" at level "$i+1$".

At level "$i+2$" insert all the new $y$'s adj. to the $x$'s at level "$i+1$".

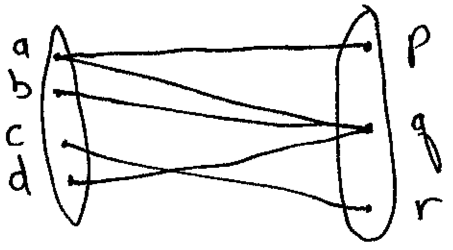
4) Repeat.

Remark: Maybe this stops because all $y$'s were covered. Then change $x_0$.

If not found $\forall x$ unmatched $\Rightarrow$ Maximum matching.

Examples:

1)



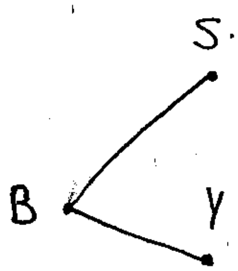

$V(\{a,b,d\}) = \{p,q\}$

$\Rightarrow$ No complete matching.

ap, bq, cr is maximum.

How to show maximum in general? (Directly tough to check).

2) ⓐ

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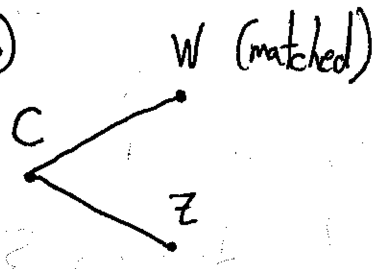


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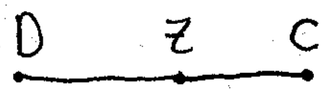
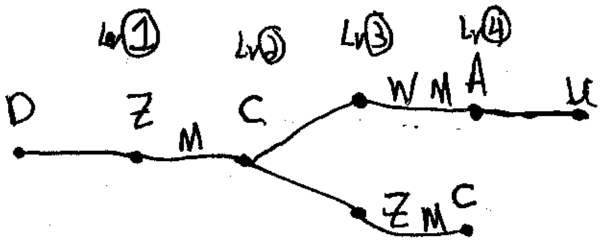


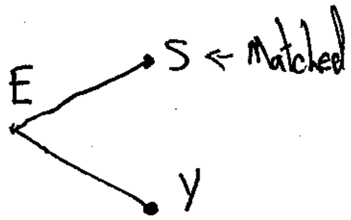

All matched

$M=\{AW\}$ $M=\{AW, BS\}$ $M=\{AW, BS, CZ\}$



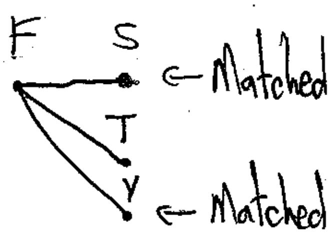

$M=\{AU, CW, DZ, BS\}$

ⓔ



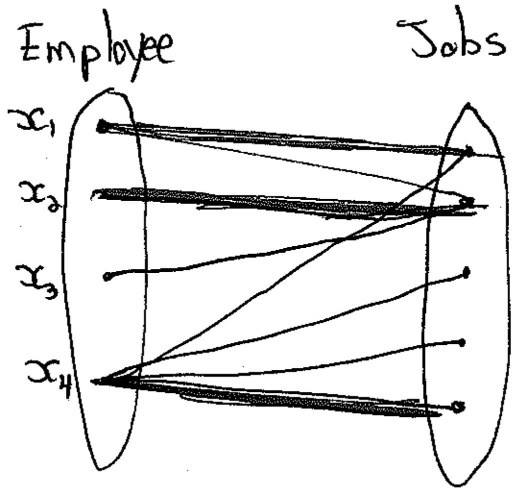

$M=\{AU, BS, CW, DZ, EY\}$

ⓕ




$M=\{AU, BS, CW, DZ, EY, FT\}$

3)




How to show it is maximum?

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**Def:** The deficiency "$d$" of a bipartite graph $G=(X \cup Y, E)$ is $d := \max_{A \subseteq X} \{ |A| - |J(A)| \}$.

$\ulcorner$ $d \geq 0$ : consider the empty set. $\lrcorner$

**Hall's thm:** $G$ has a complete matching $\Leftrightarrow$ $d = 0$.

**Thm:** The size of a maximum matching $M$ in a bipartite graph $G = (X \cup Y, E)$ is

$$|M| = |X| - d.$$

**Pf** Exercise.

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## 6. Connectivity in graphs

**Def:**
- The edge-connectivity of a graph is the minimal number of edges whose removal disconnects the graph.
- The (vertex)-connectivity of a non-complete graph — " — of vertices — " —.

- When v-connectivity $\geq k$, we say it is k-connected.

- A cut set is a minimal set of edges that disconnects the graph
- A separating set is a minimal set of vertices that disconnects the graph

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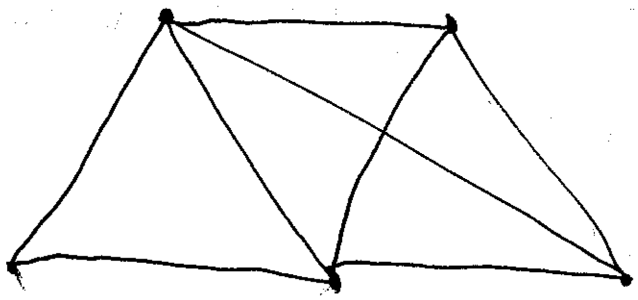
Ex:



1-conn. 2-conn.
but not 3-connected.

- Trees with $n \geq 3$ are 1-connected.

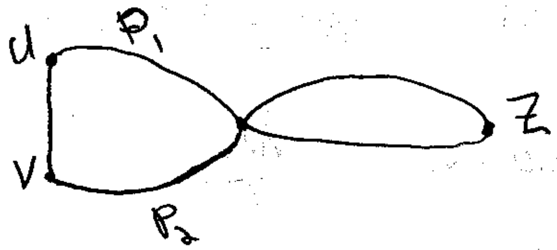
Theorem (Whitney, '27)
A connected graph with at least 3 vertices is 2-connected $\Longleftrightarrow$ $\forall$ two vertices $x, y \in V$, there is a simple cycle containing both.

Proof $\Leftarrow$) If every pair is contained in a cycle, Removing one vertex does not disconnect the graph.

$\Rightarrow$) Induction on the distance $dist(u,v)$ between two vertices.

Base case: Let $u,v$ be adjacent, and take $z \in V \setminus \{u,v\}$.

Because $G$ is 2-connected, $\exists P_1$ a path from $u$ to $z$ in $V \setminus \{v\}$, and similarly for $v$ to $z$.



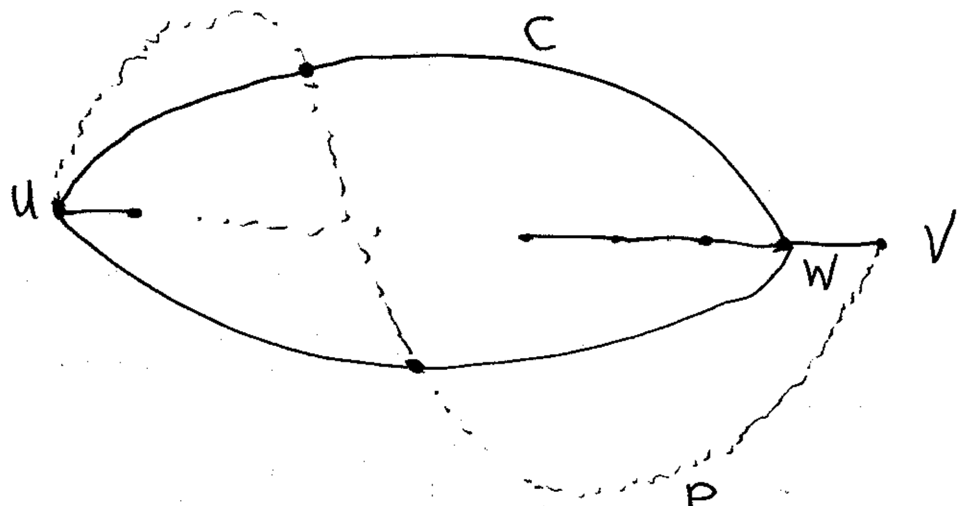

Take the first common vertex "$x$", $u \to x \to v \to u$. is a simple cycle.

Induction step: Assume it is true $\forall x,y$ such that $d(x,y) \leq k$

Take $u,v$ s.t. $d(u,v) = k+1$.

Let $P$ be a shortest path from $u$ to $v$ with last vertex $w$


- Since $dist(u,w) = k$, there is a cycle $C$ containing $u$ and $w$.
- Removing $w$ does not disconnect $u$ and $v$, hence there is a path from $u$ to $v$ without $w$.




- If C and P have no common vertices appart from u & v it is clear.
- Else:
  - If P has only common vertices on one "side" of C, pick the "otherside" of C and then come back to u with P.
  - Else, P has common vertices on both sides of C. Let $C_1$ be the first side that P touches from V to u. (say at vertex $\dot{t}$) and $C_2$ the other side. Then $u \xrightarrow{C_2} W - V \xrightarrow{P} t \xrightarrow{C_1} u$ is a simple cycle.

$\boxtimes$

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<u>Theorem</u> (Menger, 1927)

The maximum # of vertex disjoint paths connecting two distinct nonadjacent vertices v and w of a graph
= min. # of vertices in a vw-separating set.

**Corollary** A graph G with at least k+1 vertices is k-connected $\iff$ any two vertices of G are connected by at least k vertex-disjoint paths.

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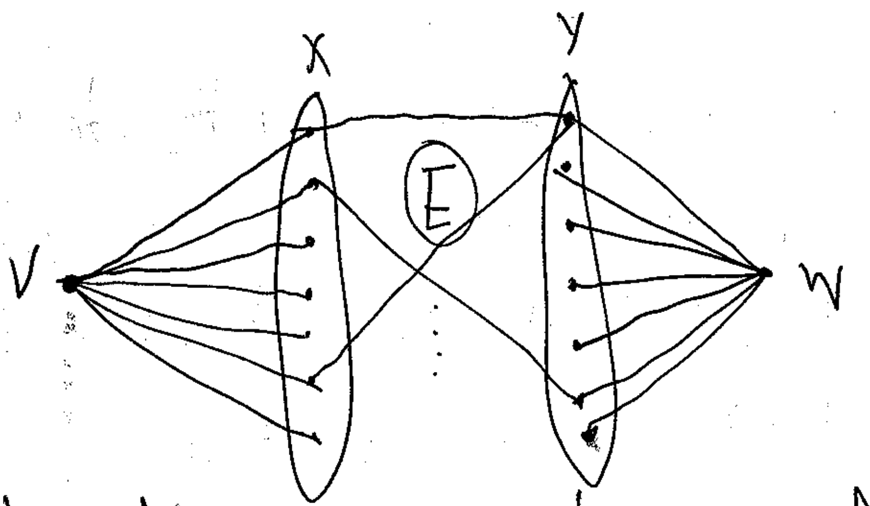
**Thm:** Menger's Theorem $\Rightarrow$ Hall's theorem

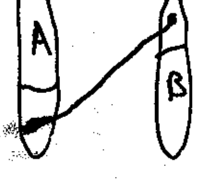
**Proof:** Let $G = (X \cup Y, E)$ be such that $|J(A)| \geq |A|$, $\forall A \subseteq X$. We have to show that it has a complete matching.

Create $G'$ as follows:




A complete matching in G exists $\iff$ # of vertex disjoint paths from v to w $=$ # vertices in X (i.e. $|X|$)

We show that every vw-separating set S has $|X|$ vertices.
$S = A \cup B$, $A \subseteq X$ and $B \subseteq Y$. Since S is vw-sep.




$\leftarrow$ Can not happen. $\Rightarrow V(X \setminus A) \subseteq B$.

$\Rightarrow |X \setminus A| \leq |V(X \setminus A)| \leq |B|$ $\Rightarrow |S| = |A| + |B| \geq |X|$ $\boxtimes$