Discrete Mathematics 1

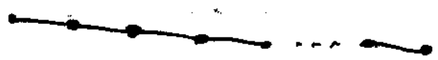
1. Basic notions (continued)

Lemma: Let G_1 and G_2 ~~be two isomorphic graphs~~. Their incidence matrices M_{G_1}, M_{G_2} are equal, up to a permutation of the rows and columns.

Def: (Graph morphism) Let G and H be graphs and $\varphi: V_G \xrightarrow{\text{fct}} V_H$. The fct φ is a graph morphism if

$$(u,v) \in E_G \Rightarrow (\varphi(u), \varphi(v)) \in E_H.$$

- A "chain" is therefore the image of a chain graph



under a graph morphism.

- edge-simple: injective on E_H
- vertex-simple: injective $\varphi: V_G \to V_H$.

- A "cycle" is similarly, the image of a cycle graph.
  - edge-simple: injective on E_V
  - vertex-simple: injective on $V_G \to V_H$

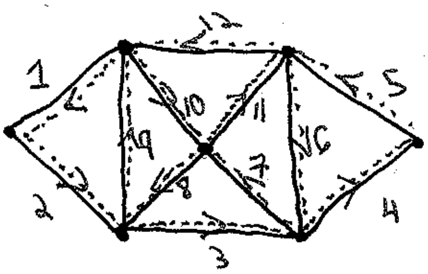
2. Cycles in Graphs

Def: (Eulerian cycle)
Let G be a graph (multi graph and loops allowed). An Eulerian cycle in G is a cycle that includes every edge of G exactly once.
Equivalently, it is the image of a graph morphism

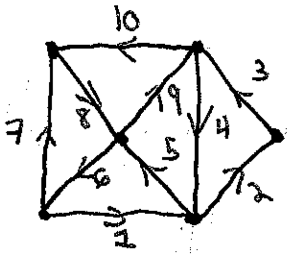
$$\varphi : C_k \to G \quad \text{which is surjective and injective on the edge set.}$$

- An Eulerian path/chain is defined similarly using a chain.
- A graph is (semi-) Eulerian if it contains an Eulerian cycle (chain).

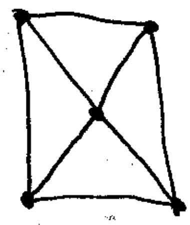
Ex:



Eulerian Graph



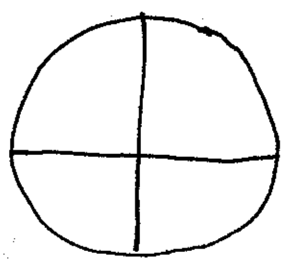
Semi-Eulerian

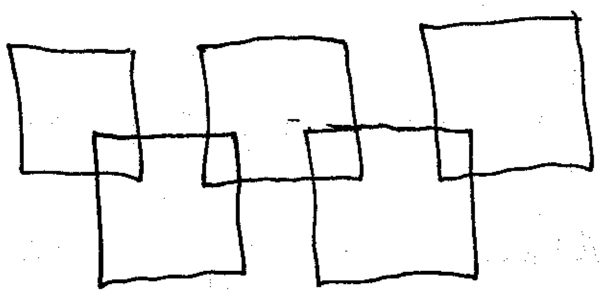


?

non-Eulerian

Is it possible to draw the following without lifting the pencil?





Can we find necessary & sufficient conditions for a graph to be Eulerian?

<u>Lemma</u>: If G is a graph such that $\deg(v) \geq 2$ $\forall v \in V$, then G contains a cycle.

<u>Proof</u>: If G has loops or multiedges, we are done.
- Suppose G is simple.

Take $v \in V$ and construct a chain $v - v_1 - v_2 - \dots$ inductively. It is always possible to continue it such that $v_{i-1} \neq v_{i+1}$ by the hypothesis.

Since G is finite, by the Pigeonhole Principle, when the chain is long enough a vertex will repeat.

If v_k is such a vertex the chain between two occurrences of v_k is the cycle. ☒

<u>Theorem</u>: (Euler, 1736)

A connected graph G is Eulerian if and only if the degree of each vertex of G is even.

<u>Proof</u>: $\Rightarrow$) Let C be a Eulerian cycle of G.

Whenever C passes through a vertex, there is a contribution of 2 to the degree of that vertex.

Since edges occur exactly once in C, each vertex must have even degree.

$\Leftarrow$ By induction on E

Since G is connected, every vertex has degree ≥ 2. By the lemma, G contains a cycle C.

Base case: $E \setminus C = \emptyset$. C is an Eulerian cycle. (it is vertex-simple).

Induction step: If $E \setminus C \neq \emptyset$.

Remove edges in C to form G' with fewer edges and $\forall v \in G'$, the vertices still have even degree.

- G' may be disconnected (i.e. singleton vertices).

- By induction hypothesis, each component of G' has an Eulerian cycle.
- By connectedness of G each component of G' shares a vertex of C.
- The Eulerian cycle is formed by going through C until we reach a nonisolated vertex of G', we concatenate the Eulerian cycle of the component and continue...

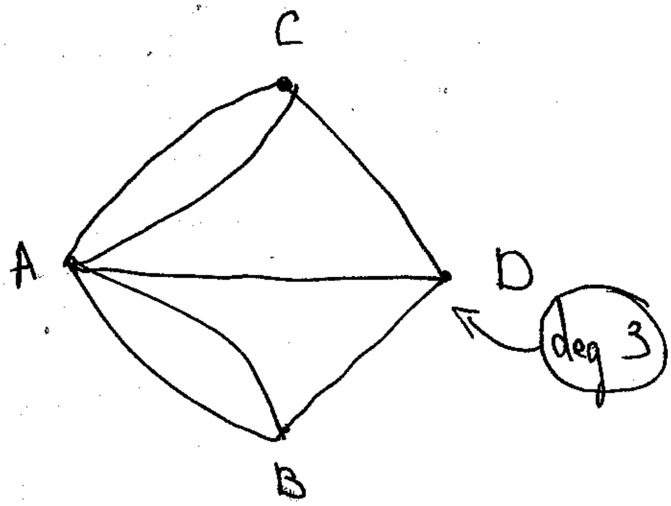
$\boxtimes$

Corollary: A connected graph is Eulerian if and only if its set of edges can be split up into disjoint cycles.

Corollary: A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.

Note: By the HandShake Lemma, a graph cannot have exactly one vertex of odd degree.

So, is the graph



Eulerian? Semi-Eulerian?

We can ask a similar question for embedding vertex-simple cycles going through all vertices.

Def: · An Hamiltonian cycle in G is a vertex-simple cycle that goes through every vertex.

· A graph with an Hamiltonian cycle is called Hamiltonian.

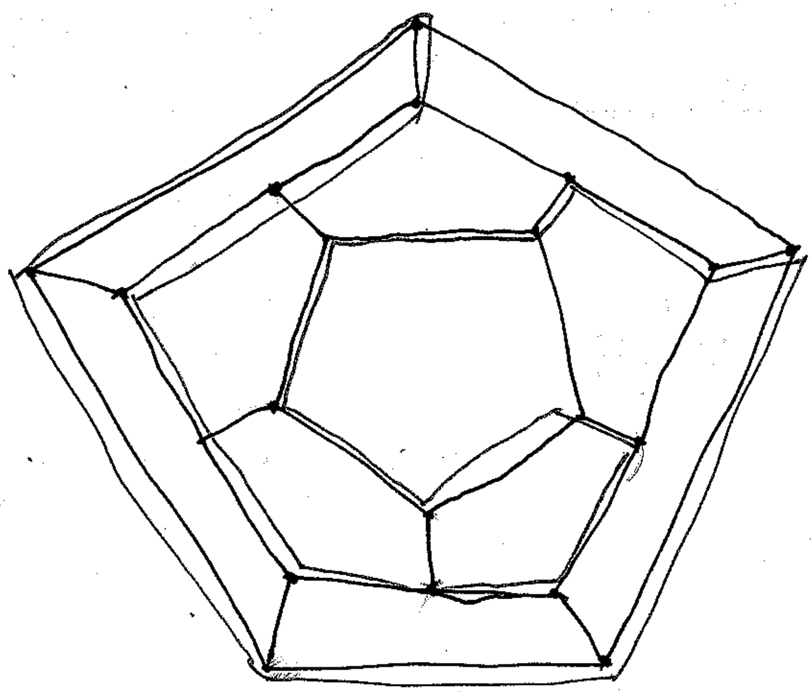
BIG DEAL QUESTION: Find necessary and sufficient conditions on Graphs to be hamiltonian.

→ There is no known simple characterization.

→ Many families are studied and known to be / not to be.

→ C_n and K_n $n \geq 3$.

<u>Icosian Game (1857)</u>: (Hamilton)



<u>Thm</u>: The graph Q_n is hamiltonian, $\forall n \geq 2$.

<u>Proof</u>: If $n=2$, $Q_2 \cong C_4$ → Hamiltonian.

- Assume Q_n to be Hamiltonian and $n \geq 2$.
- Partition the vertices of $Q_{n+1} = A \sqcup B$

where $A = \{ v \in Q_{n+1} \mid v \text{ starts with a } 0 \}$

$B = \{ v \in Q_{n+1} \mid v \text{ — '' — } 1 \}$

The induced subgraphs Q_{n+1}^{A} and Q_{n+1}^{B} are isomorphic to Q_n.

$\Rightarrow$ They are Hamiltonian.

- We can assume that the Hamiltonian cycles in A and B are the same up to changing the first bit $0 \leftrightarrow 1$.
- Take an edge in H_A (the hamiltonian cycle in A)

$$e = (v_1, v_2) \in H_A \qquad f = (v_1', v_2')$$

$$v_1 \xleftrightarrow{0 \leftrightarrow 1} v_1' \qquad v_2 \xleftrightarrow{0 \leftrightarrow 1} v_2'$$

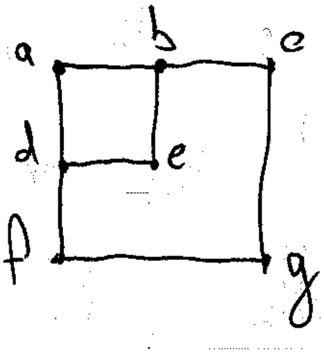
- Remove "e" from H_A and "f" from H_B.
- Concatenate H_A (from v_2 to v_1) with (v_1, v_1') and then H_B (from v_1' to v_2') followed by (v_2' to v_2).

This is a Hamiltonian cycle

$\boxtimes$

Trivially, to be Hamiltonian, $\deg(v) \geq 2 \ \forall v \in V$.

But not sufficient:



Theorem (Ore's Theorem) (1960).

Let G be a finite and simple graph of order $n \geq 3$.

If $\deg(v) + \deg(w) \geq n$ $\forall$ $v, w \in G$ non-adjacent,

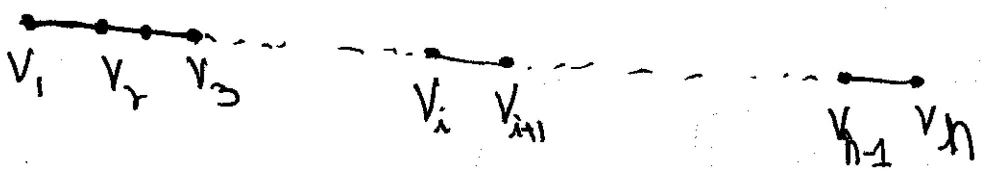
then G is Hamitonian.

Proof: By contradiction assume that G satisfies the degree condition but is not Hamiltonian.

Further, we can add edges to G until adding any edge will create an Hamiltonian cycle.

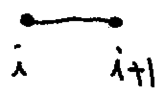
- This does not influence the degree condition.

Hence, pick two non-adjacent vertices v_1, v_n, since adding an edge between them makes an Hamiltonian cycles, there is an Hamiltonian chain between v_1 and v_n:



So $\deg(v_1) + \deg(v_n) \geq n$

Consider $1 \leq i \leq n-1$, for the edge



v_1 can be connected to v_{i+1} or v_n to v_i

BUT not both.

Otherwise

$$v_1 - v_2 - \cdots - v_i - v_n - v_{n-1} - \cdots - v_{i+1} - v_1$$

Is an Hamiltonian cycle.

Hence we can have at most $n-1$ edges coming out of v_1 or v_n.

That is $\deg(v_1) + \deg(v_n) \leq n-1$ ↯ ☒

<u>Corollary</u> (Dirac's Theorem, 1952)

If G is finite, simple, of order $n \geq 3$, and $\deg(v) \geq n/2$ $\forall v \in V$, then G is hamiltonian.

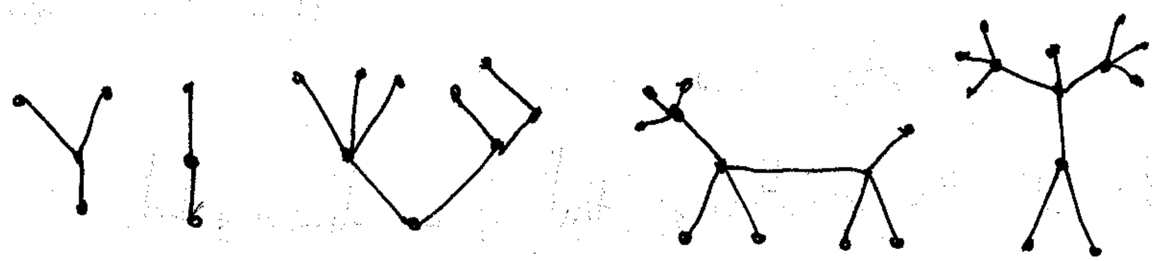
Pf| Since $\deg(v) + \deg(w) \geq n$ $\forall v, w \in V$. ☒

<u>3. Acyclic graphs: Forests & Trees</u>

<u>Def:</u> • A <u>forest</u> is a graph that contains no cycles (hence no loops and no multiedges).

• A connected forest is a <u>tree</u>.

<u>Ex:</u>



Theorem: (Characterizations of Trees)

Let T be a graph of order n. The following statements are equivalent:

i) T is a tree.

ii) T contains no cycles, and has n-1 edges.

iii) T is connected, and has n-1 edges.

iv) T is connected, and each edge is a bridge.

v) any two vertices of T are connected by exactly one path.

vi) T contains no cycles, and adding any new edge creates exactly one cycle.

Def: A bridge is an edge of a (connected) graph whose removal disconnects the graph.

Proof: For $n=1$, all statements are trivially equivalent.

Assume $n \geq 2$.

i) ⇒ ii) Since T is a tree, removing an edge disconnects the two vertices. and creates two connected components (trees). By induction on n, the number is $|V_1|-1+|V_2|-1$ and adding back the removed edge gives $n-1$.

ii) ⇒ iii) Assume otherwise that it is disconnected.

That is T is a forest, hence each piece has 1 more vertex than edges.

$$\Rightarrow \quad |V| \geq |E| + 2 \quad \lightning \text{ with } |E| = n-1.$$

<u>iii) $\Rightarrow$ iv)</u>: Removing an edge gives a graph with n vertices and $n-2$ edges.

<u>Claim</u>: Such a graph is disconnected (Exercise)

<u>iv) $\Rightarrow$ v)</u>: Since T is connected there is at least one path. If there are two disjoint chains $\Rightarrow$ T contains a cycle (chain) (Ex.10 Sheet 9)

$\Rightarrow$ Not all edges are bridges. $\lightning$

<u>v) $\Rightarrow$ vi)</u>: • If T has a cycle there would be two paths. $\lightning$

• Since T is connected, adding an edge creates a cycle.

<u>Claim</u>: This creates only 1 cycles. (Exercice).

<u>vi) $\Rightarrow$ i)</u>: Suppose that T is disconnected, then adding an edge between two vertices of 2 diff. connected comp. does <u>not</u> create a cycle. $\lightning$

$\boxtimes$

<u>Corollary</u>: If G is a forest w/ n vertices and k components, then G has $n-k$ edges.

Theorem: (Cayley, 1889)

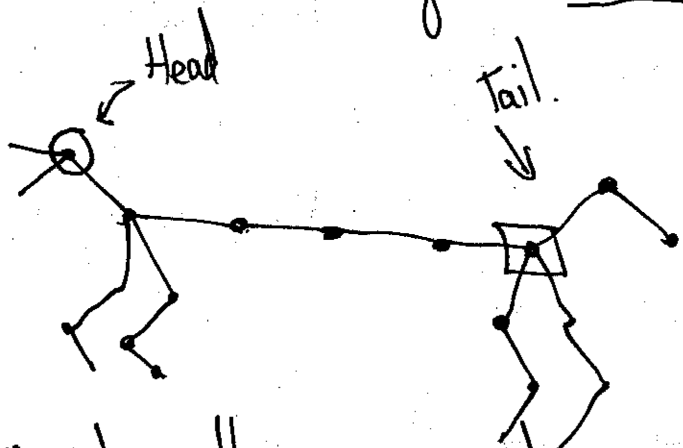
The number of labeled trees on n vertices is n^{n-2}.

Pf: (Joyal, '81)

Let T_n denote this number.

Let $\mathbb{V}_n$ denote the set of doubly rooted trees, labeled trees with 2 distinguished nodes "Head" and "Tail".

We can call such a thing a "vertebrae":



Head & Tail can be the same node.

Hence $|\mathbb{V}_n| = n^2 \cdot T_n$

Now, let $F_n = \{ f : [n] \to [n] \mid f \text{ function} \}$

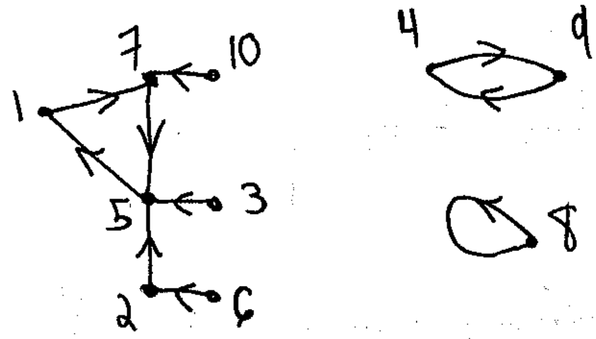
$\Rightarrow |F_n| = n^n$.

For each $f \in F_n$, draw an oriented graph:

We will create a vertebrae.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 5 & 5 & 9 & 1 & 2 & 5 & 8 & 4 & 7 \end{pmatrix}$$

G_f



Each component has k vertices and k edges.

$\Rightarrow$ By Char. of trees vi) there is a cycle (which is unique).

Since "out degree" $= 1$ the cycle is <u>oriented</u>.

Let $M = \{ v \in G_f \mid v$ belongs to an oriented cycle $\}$.

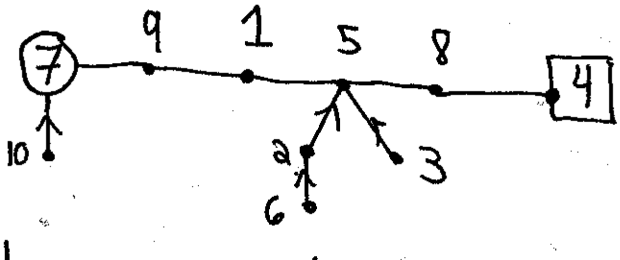
f is a permutation of M.

$$f|_M = \begin{pmatrix} a & b & c & \cdots & z \\ f(a) & f(b) & & & f(z) \end{pmatrix} \qquad a < b < \ldots < z.$$

$$f(a) := \text{Head} \qquad = \begin{pmatrix} 1 & 4 & 5 & 7 & 8 & 9 \\ 7 & 9 & 1 & 5 & 8 & 4 \end{pmatrix}$$

$f(z) :=$ Tail

T_f :



<u>Vertebrae.</u>

Reverse: Write $f|_M$, then orient from i to the spine.

$$\Rightarrow |\mathcal{V}_n| = n^n = n^2 T_n \iff T_n = n^{n-2}.$$

$\boxtimes$

→ There are many other proofs of Cayley's Theorem.

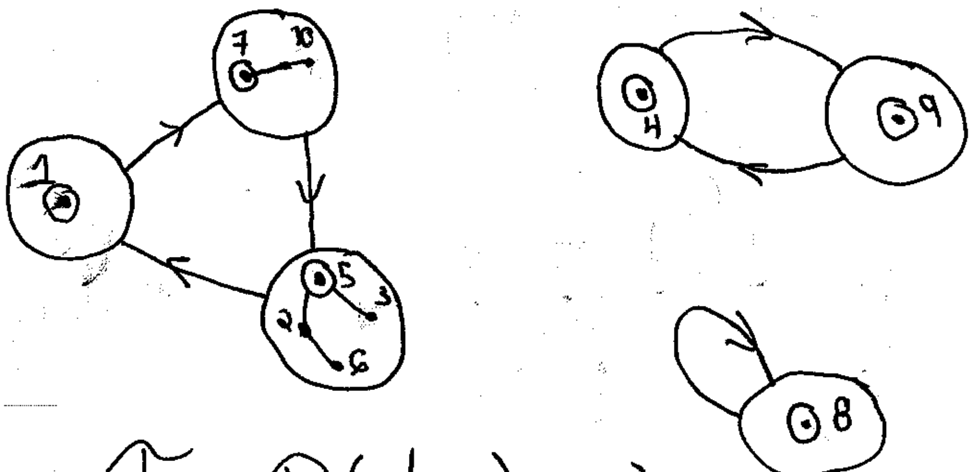
This proof hides the following:

Let $\mathcal{V}_n$ be the generating function for V_n.

- A vertebrae is given by a sequence of rooted trees. (from head to tail).
  So if $\mathcal{R}(x)$ is the generating function for rooted trees,

$$\mathcal{V}(x) = \mathcal{R}(x) + \left(\mathcal{R}(x)\right)^2 + \left(\mathcal{R}(x)\right)^3 + \cdots$$

- An endofunction $f : [n] \to [n]$ is given by a permutation of rooted trees:



That is $\quad \mathcal{F}(x) = \mathcal{P}\left(\mathcal{A}(x)\right) \cong \mathcal{V}(x) = \sum n^n \dfrac{x^n}{n!}$

$$\mathcal{P}\left(\mathcal{A}(x)\right) \underset{\text{is}}{=}\ \mathcal{L}\left(\mathcal{A}(x)\right)$$