Supplementary Exercises

Discrete Mathematics I - SoSe17

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These exercises cover the material seen in the first half of the semester.

Week 1

Numbers, Equinumerousity, Addition-Multiplication principles, Bijective Proof

Problem 1

True or False: If $2^X = 2^Y$ holds for two sets X and Y, then $X = Y$.

Problem 2

Let $M \subseteq \mathbb{R}$ be a set of real numbers such that any nonempty subset of M has a least member and a greatest member. Prove that M is necessarily finite.

Problem 3

At a particular company, any valid password starts with six lower case letters followed by two digits.

a) How many valid passwords are there?

b) How many valid passwords do not start with "a"?

c) How many valid passwords do not end with "88"?

Problem 4

Give a bijective proof of the following identity

$$\sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n},$$

where $n \geq 1$.

Week 2

Bijective Proof, Binomial Theorem, Inclusion-Exclusion Principle and Pigeonhole Principle

Problem 5

A basket of fruit is being arranged out of apples, bananas and oranges. What is the smallest number of pieces of fruit that should be put in the basket in order to guarrantee that either there are at least 8 apples or at least 6 bananas or at least 9 oranges?

Problem 6

Find the number of integers between 1 and 1000 inclusive, which are divisible by none of 5, 6 and 8.

Problem 7

Prove the binomial theorem by induction on n:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Problem 8

Let N be an n-element set and M be an m-element set. Define a bijection between the set of all mappings $f : N \to M$ and the n-fold cartesian product M^n.

Week 3

The Twelvefold Way: Integer partitions, Set-partitions, compositions, Stirling numbers, Bell numbers

Problem 9

Ten people split up into five groups of two each. In how many ways can this be done?

Problem 10

A box is filled with three blue socks, three red socks, and four green socks. Eight socks are pulled out, one at a time. In how many ways can this be done? (Socks of the same color are undistinguishable.)

Problem 11

Let $1 \leq k < n$. Give a combinatorial proof that among all the 2^{n-1} compositions of n, the part k occurs a total of $(n-k+3)2^{n-k-2}$ times. For instance, if $n = 4$, and $k = 2$, then the part 2 appears once in 2+1+1, 1+2+1, 1+1+2 and twice in 2+2, for a total of five times.

Problem 12

Find the number of compositions of 35 into seven odd parts.

Week 4

Multinomial coefficients, weak compositions, Ferrers diagrams, Solving recurrence relations

Problem 13

Prove that

$$\sum_{a_1+a_2+a_3=n} \binom{n}{a_1, a_2, a_3} = 3^n.$$

Problem 14

Prove that

$$\frac{1}{\sqrt{1-4x}} = \sum_{n\geq 0} \binom{2n}{n} x^n.$$

Problem 15

Prove that the number of weak compositions of n into k parts is $\binom{n-1}{k-1}$.

Problem 16

Give the general solution of the recurrence relation:

$$F_n = 3F_{n-1} + 5F_{n-2} - 7F_{n-3}.$$

Week 5

Solving Rec. Rel. (Symbolic Differenciation and undetermined Coeff. Methods) Formal Power Series: Definitions and Operations. Generating Functions: basic examples.

Problem 17 (updated)

Find a closed form for the recurrence $R_n = 5R_{n-1} + 29R_{n-2} - 105R_{n-3}$, where $R_0 = 0$, $R_1 = 1$, and $R_2 = 1$ using **generating functions**.

Problem 18 (updated)

Find a closed form for the recurrence $R_n = 5R_{n-1} + 29R_{n-2} - 105R_{n-3}$, where $R_0 = 0$, $R_1 = 1$, and $R_2 = 1$ using the **characteristic polynomial**.

Problem 19

Find a closed form for the recurrence $R_n = 5R_{n-1} + 4^n$, where $R_0 = 1$ using **symbolic differentiation**.

Problem 20

Find the number of unordered 15 letter words from the alphabet $\{a, b, c, d, e\}$ that satisfy the following:

a) If 'a' is used, then it is used seven times,

b) The number of times 'b' appears is a multiple of four,

c) 'c' appears at least two times,

d) 'd' appears no more than five times,

e) 'e' appears a prime number of times.

What if we require all properties to be true?

Week 6

Applications of Generating Functions to enumeration. Recurrence relations. Catalan numbers. Exponential Generating Functions.

Problem 21

Find the number of ordered 15 letter words from the alphabet $\{a, b, c, d, e\}$ that satisfy the following:

a) If 'a' is used, then it is used seven times,

b) The number of times 'b' appears is a multiple of four,

c) 'c' appears at least two times,

d) 'd' appears no more than five times,

e) 'e' appears a prime number of times.

What if we require all properties to be true?

Problem 22

Let $f(n)$ be the number of subsets of $[n]$ in which the difference between any two elements is at least three. Find the generating function of $f(n)$.

Problem 23

There are two candidates A and B in an election. Each receives n votes. What is the probability that A will never trail B during the count of votes?

Problem 24

Determine the generating function for the number H_n of nonnegative integral solutions of

$$x_1 + x_2 + x_3 + x_4 = n.$$

Week 7

Posets: Definitions and properties, constructions, examples, Lattices

Problem 25

Draw the Hasse diagram of the principal lower ideal of the Young's lattice generated by the partition $(5, 4, 2, 1, 1)$.

(The diagram has 88 elements, you don't need to draw the bottom where it has less that 5 elements)

Problem 26

Give the values of the Möbius function on the poset $(\{n \in \mathbb{N} \setminus \{0\} : n|24\}, |)$ (that is, the poset of natural numbers that divide 24 ordered by division).

Problem 27

Let (S, τ) be a topological space and order the collection of open sets τ by inclusion to get a poset $(\tau, \subseteq)$.

a) Define compactness of subsets of S in terms of the poset $(\tau, \subseteq)$.

b) Define connectedness of S in terms of the poset $(\tau, \subseteq)$.

Problem 28

Show that for every poset $(P, \leq)$ there exists an injective function $f : P \to 2^P$ such that $x \leq y$ if and only if $f(x) \subseteq f(y)$. This function is an *embedding* of P into a Boolean lattice.

Harder Problems

Problem A

Give a combinatorial proof of the following identity.

$$\sum_{i=0}^{n} \binom{2i}{i} \binom{2(n-i)}{n-i} = 4^n.$$

Problem B

Find a closed form for the recurrence $R_n = 8R_{n-1} - 20R_{n-2} + 32R_{n-3} - 64R_{n-4} + (-1)^n + 4^n + n^2 - \sin(n\pi)$, where $R_0 = R_1 = R_2 = R_3 = 1$.

Problem C (updated)

Let c_n be the number of functions $f : [n] \to [n]$ such that $f(i) \leq f(j)$ for all $i < j \in [n]$. Find a closed formula for c_n.

Problem D

Determine the generating function for the number H_n of nonnegative integral solutions of

$$2x_1 + 5x_2 + x_3 + 7x_4 = n.$$

Problem E

Find a poset P for which there is a bijection $f : P \to P$ such that $x \leq y$ if and only if $f(x) \geq f(y)$ (i.e. P is self-dual), but for which there is *no* such bijection f satisfying $f(f(x)) = x$ for all $x \in P$.