Exercise Sheet 8 (Partial Solutions)

Discrete Mathematics I - SoSe17

Lecturer Jean-Philippe Labbé

Tutors Johanna Steinmeyer and Patrick Morris

Due date 14 June 2017 -- 16:00

You should solve all of the exercises below, and select three to four solutions to be submitted and graded. We encourage you to submit in pairs, please remember to

i) indicate the author of **each** individual solution,
ii) the **name of both team members** on the cover sheet,
iii) **read carefully** the question.

Problem 1

Let P be a locally finite poset and $x < y \in P$. Prove that

$$\mu(x,y) = c_0 - c_1 + c_2 - c_3 + \cdots ,$$

where c_i is the number of chains of length i from x to y. (Therefore $c_0 = 0$, and $c_1 = 1$.)

Let $\eta := \zeta - \delta$. Then

$$\begin{aligned}\mu = \zeta^{-1} &= (\delta + \eta)^{-1} \\ &= \sum_{i \geq 0} (-\eta)^i = \delta - \eta + \eta^2 - \eta^3 + \ldots\end{aligned}$$

By a proposition in class, the η^i function counts the chains from any x to any y and we get the result.

Problem 2

Find a lattice L such that the following is **not true**

$$\text{if } x \leq z, \text{ then } x \vee (y \wedge z) = (x \vee y) \wedge z, \text{ for all } y \in L$$

Problem 3

A *topological space* on a set S may be defined by a collection of *open sets* containing S and $\varnothing$ that is closed under arbitrary unions and finite intersections. A topology is T_0 if for any two elements x and y, there is an open set containing x but not y, or an open set containing y but not x. Show that if P is a finite set, a T_0-topology defines a partial order, and conversely, a partial order defines a T_0-topology.

Problem 4

A poset P has the *fixed-point property* if for every order preserving function $f : P \to P$, there exists a point $x \in P$, such that $f(x) = x$. Show that if P is finite and contains a greatest element $\hat{1}$, then P has the fixed-point property.

By induction on the size of P. If $n = 1$ it is trivially true. Assume that it is true for all size strictly smaller than n. If $f(\hat{1}) = \hat{1}$, the function has a fixed point. Else, $\hat{1}$ is sent to an element strictly smaller than it, say x. Then, the lower ideal $\langle x \rangle$ generated by x has strictly less elements than P and f restricted to $\langle x \rangle$ is an order preserving map from the ideal to itself. By induction it has a fixed point since x is the greatest element and it is finite. Thus f also has a fixed point.

Problem 5

Let G be a finite group and $L(G)$ be the lattice of subgroups of G (ordering by inclusion). Show that the function π defined from $L(G)$ to the lattice of partitions of the ground set G (ordered by refinement) sending a subgroup H to the partition $\pi(H)$ of G by the cosets of H is a lattice homomorphism. That is: if H and K are subgroups of G, then $\pi(H \cap K) = \pi(H) \wedge \pi(K)$ and $\pi(H \vee K) = \pi(H) \vee \pi(K)$.

Problem 6

Define a binary operation on the set $Int(P)$ of intervals of a locally finite poset P by

$$[a, b] \times [c, d] := \begin{cases} [a, d], & \text{if } b = c \\ 0, & \text{otherwise.} \end{cases}$$

a) Show that $(Int(P), \times)$ is a semigroup.

b) Show that the $\mathbb{C}$-algebra over this semigroup is isomorphic to the incidence algebra $\mathcal{I}(P)$ over $\mathbb{C}$.

Problem 7

Consider the set P_n of set-partitions of $[n]$ with the refinement order $\leq$. (For example $\{1, 4\}\{2, 3\}\{5\}\{6\} \leq \{1, 4, 6\}\{2, 3, 5\}$. Show that $(P_n, \leq)$ forms a lattice.

Problem 8

Give a formula for the Möbius function of the lattice $(P_n, \leq)$ of Problem 7.

Problem 9

Prove that the Möbius function of the Young lattice is:

$$\mu(p, q) = \begin{cases} (-1)^{|p|-|q|}, & \text{if the skew diagram } p/q \text{ is a disconnected union of squares,} \\ 0, & \text{otherwise.} \end{cases}$$

Problem 10

The *dimension* of a poset P is the minimum number of linear orders of the vertex set of P so that the intersection of these linear orders is precisely the poset P. Find a map from permutations to posets of dimension two. Is this map injective?

Write a permutation in two line notation $\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$. Send the permutation π to the poset obtained by the intersection of the two liner orders. It is not injective if we consider the posets to be unlabeled (consider a 3-cycle and the bottom-top row-switch, they give the same unlabeled poset.)