Exercise Sheet 8

Discrete Mathematics I - SoSe17

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You should solve all of the exercises below, and select three to four solutions to be submitted and graded. We encourage you to submit in pairs, please remember to

i) indicate the author of **each** individual solution,
ii) the **name of both team members** on the cover sheet,
iii) **read carefully** the question.

Problem 1

Let P be a locally finite poset and $x < y \in P$. Prove that

$$\mu(x, y) = c_0 - c_1 + c_2 - c_3 + \cdots ,$$

where c_i is the number of chains of length i from x to y. (Therefore $c_0 = 0$, and $c_1 = 1$.)

Problem 2

Find a lattice L such that the following is **not true**

$$\text{if } x \leq z, \text{ then } x \vee (y \wedge z) = (x \vee y) \wedge z, \text{ for all } y \in L$$

Problem 3

A *topological space* on a set S may be defined by a collection of *open sets* containing S and $\varnothing$ that is closed under arbitrary unions and finite intersections. A topology is T_0 if for any two elements x and y, there is an open set containing x but not y, or an open set containing y but not x. Show that if P is a finite set, a T_0-topology defines a partial order, and conversely, a partial order defines a T_0-topology.

Problem 4

A poset P has the *fixed-point property* if for every order preserving function $f : P \to P$, there exists a point $x \in P$, such that $f(x) = x$. Show that if P is finite and contains a greatest element $\hat{1}$, then P has the fixed-point property.

Problem 5

Let G be a finite group and $L(G)$ be the lattice of subgroups of G (ordering by inclusion). Show that the function π defined from $L(G)$ to the lattice of partitions of the ground set G (ordered by refinement) sending a subgroup H to the partition $\pi(H)$ of G by the cosets of H is a lattice homomorphism. That is: if H and K are subgroups of G, then $\pi(H \cap K) = \pi(H) \wedge \pi(K)$ and $\pi(H \vee K) = \pi(H) \vee \pi(K)$.

Problem 6

Define a binary operation on the set $Int(P)$ of intervals of a locally finite poset P by

$$[a,b] \times [c,d] := \begin{cases} [a,d], & \text{if } b = c \\ 0, & \text{otherwise.} \end{cases}$$

a) Show that $(Int(P), \times)$ is a semigroup.

b) Show that the $\mathbb{C}$-algebra over this semigroup is isomorphic to the incidence algebra $\mathcal{I}(P)$ over $\mathbb{C}$.

Problem 7

Consider the set P_n of set-partitions of $[n]$ with the refinement order $\leq$. (For example $\{1,4\}\{2,3\}\{5\}\{6\} \leq \{1,4,6\}\{2,3,5\}$. Show that $(P_n, \leq)$ forms a lattice.

Problem 8

Give a formula for the Möbius function of the lattice $(P_n, \leq)$ of Problem 7.

Problem 9

Prove that the Möbius function of the Young lattice is:

$$\mu(p,q) = \begin{cases} (-1)^{|p|-|q|}, & \text{if the skew diagram } p/q \text{ is a disconnected union of squares,} \\ 0, & \text{otherwise.} \end{cases}$$

Problem 10

The *dimension* of a poset P is the minimum number of linear orders of the vertex set of P so that the intersection of these linear orders is precisely the poset P. Find a map from permutations to posets of dimension two. Is this map injective?