Problem 1 3

Let T_n be the number of ways to arrange n books on two bookshelves so that each shelf receives at least one book.

a) The exponential generating function (EGF) $T^{(e)}(x)$ of T_n is obtained as follows. Let A_n and B_n denote the ways of placing n books on the first and second shelves respectively. As the shelves are identical, and the problem for one shelf reduces to permutations, we can write $A_n = B_n = n!$ for $n \geq 1$ while $A_0 = B_0 = 0$. The respective EGFs are:

$$A^{(e)}(x) = B^{(e)}(x) = 1!\frac{x}{1!} + 2!\frac{x^2}{2!} + 3!\frac{x^3}{3!} + \ldots = 1 + x^2 + x^3 + \ldots = \frac{x}{1-x}$$

$T^{(e)}(x)$ is given by the product:

$$T^{(e)}(x) = A^{(e)}(x)B^{(e)}(x) = \left(\frac{x}{1-x}\right) = \frac{x^2}{(1-x)^2}$$

b) To find T_n, recall that $T^{(e)}(x) = \sum_{n\geq 0} T_n \cdot \left(\frac{x^n}{n!}\right)$, so we rewrite the expression in part a).

$$T^{(e)}(x) = x^2 \cdot \left(\frac{1}{1-x}\right)' = x^2 \sum_{i\geq 0}(i+1)x^i \overset{n=i+2}{=} \sum_{n\geq 2}(n-1)x^n = \sum_{n\geq 2} n!(n-1)\frac{x^n}{n!}$$

Yielding $T_n = n!(n-1)$ for $n \geq 2$, and $T_0 = T_1 = 0$.

c) Let C_n and D_n be the number of ways of arranging the books if the shelves are circular and the arrangements are cyclical. For each shelf, we place an arbitrary book first, and the total number of arrangements is the same as the number of ways of arranging the $n-1$ books remaining. This gives $C_n = D_n = (n-1)!$ for $n \geq 1$ and exponential generating functions

$$C^{(e)}(x) = D^{(e)}(x) = \sum_{n\geq 1}(n-1)!\frac{x^n}{n!} = \log\left(\frac{1}{1-x}\right)$$

In this case, the new product is given by the formal power series product definition:

$$T^{(e)}(x) = C^{(e)}(x) \cdot D^{(e)}(x) = \log\left(\frac{1}{1-x}\right)^2 = \sum_{n\geq 1}\left(\sum_{i=0}^{n}\binom{n}{i}(i-1)!(n-i-1)!\right)\frac{x^n}{n!}$$

The coefficients T_n are then given by $T_0 = T_1 = 0$ and for $n \geq 2$:

$$T_n = \sum_{i=1}^{n}\binom{n}{i}(i-1)!(n-i-1)! = \sum_{i=1}^{n-1}\frac{n!(i-1)!(n-i-1)!}{(n-i)!i!} = \sum_{i=1}^{n-1}\frac{n!}{(n-i)i}$$

Discrete Mathematics I - Exe

Problem 7. (Claudia)

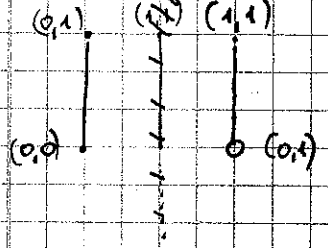
3

Team

Disproof by counterexample:

Let $S := (\{0\} \times [0,1]) \cup (\{1\} \times (0,1])$ and (*) $(x,y) \le (a,b) :\Leftrightarrow (x=a) \wedge (y \le b)$

$\forall\, (x,y), (a,b) \in S$



Then $(0,0)$ is the unique minimal element, but not a least one (we can't compare it to $(1,1)$ for example.

To show: (*) is an order relation

① reflexivity: $\forall (x,y) \in S \quad (x,y) \le (x,y) \Leftrightarrow (x=x) \wedge (y \le y)$ ✓

② antisymmetry: $\forall (x,y),(a,b) \in S \quad (x,y) \le (a,b) \wedge (x,y) \ge (a,b) \Leftrightarrow (x=a) \wedge (y \le b) \wedge (y$

$\Leftrightarrow (x=a) \wedge (y=b) \Leftrightarrow (x,y) = (a,b)$ ✓

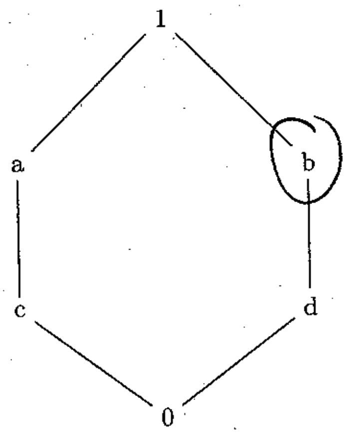
③ transitivity: $\forall (x,y),(a,b),(c,d) \in S \quad (x,y) \le (a,b) \wedge (a,b) \le (c,d)$

$\Leftrightarrow (x=a) \wedge (y \le b) \wedge (a=c) \wedge (b \le d) \Rightarrow (x=c) \wedge (y \le d)$

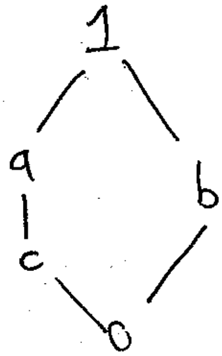
$\Leftrightarrow (x,y) \le (c,d)$ ✓

Problem 9 $\boxed{3}$ ☆

We use the following lattice L to show the distributivity axiom is not true for lattices:



← not used!



It is obvious that each pair of elements in the lattice has an infimum and a supremum.
We assume that for each elements $x, y, z \in L : x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
We choose $x = a, y = d, z = c$.
Let's determine the LHS of the equation:
$a \wedge (d \vee c) = a \wedge 1$
$= a$
Let's determine the RHS of the equation:
$(a \wedge d) \vee (a \wedge c) = 0 \vee (a \wedge c)$
$= 0 \vee c = c$
Since the RHS and the LHS are not equal, this leads us to a contradiction.

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Problem 10

[Kelvin]