Exercise Sheet 7

Discrete Mathematics I - SoSe17

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You should try to solve all of the exercises below, and select three to four solutions to be submitted and graded. We encourage you to submit in pairs, please remember to

i) indicate the author of **each** individual solution,

ii) the **name of both team members** on the cover sheet,

iii) **read carefully** the question.

Problem 1

Let T_n be the number of ways to arrange n books on two bookshelves so that each shelf receives at least one book.

a) Find the exponential generating function of T_n is closed form.

b) Give a closed formula for T_n.

c) Suppose the shelves are circular (the books on a shelve form a cycle), give the exponential generating function and a formula for T_n in closed form.

Problem 2

We have n cards. We want to split them into an even number of subsets of odd cardinality, form a line within each subset, then arrange the subsets back into a line. In how many ways can we do this?

Problem 3

For each of the following pairs, say if it is a poset. If it is a poset, say furthermore if it is totally ordered, and if it is well-ordered.

a) $(\mathbb{Z}, \leq)$, where $\leq$ is the usual order on $\mathbb{Z}$.

b) $(\mathbb{N} \setminus \{0\}, \leq)$, where $a \leq b \Leftrightarrow a = b^k$ for some $k \geq 1$.

c) $(\mathbb{N} \setminus \{0\}, R)$, where $(a, b) \in R \Leftrightarrow a \leq b + 1$ where $\leq$ is the usual order on $\mathbb{N}$.

d) $(\{1, 2, 3, 4, 6, 8, 12, 24\}, |)$, where $a|b \Leftrightarrow a$ divides b.

e) $(2^X, \leq)$, where $A \leq B \Leftrightarrow A \supseteq B$.

f) $(E, \leq)$, where $E := \{1\} \cup \{2^{2^n} : n \in \mathbb{N}$ and $a \leq b \Leftrightarrow a^k = b$ for some $k \in \mathbb{N}\}$.

Problem 4

In each case below, give an example of poset $(P, \leq)$ having that property.

a) P is countable and $(P, \leq)$ is connected but not locally finite nor total.

b) P as one maximal element but no greatest element.

c) P has no least element nor minimal elements, but a greatest element.

d) Let ℓ be the length of the longest chain of P. Every $x \in P$ is contained in a chain of length ℓ and P has a maximal chain of length $< \ell$.

Problem 5

Let $(P, \leq)$ be a finite poset. Show that P can be partitioned into $\alpha(P)$ disjoint chains, where $\alpha(P)$ is the size of the largest antichain of P.

Problem 6

Show that every subset of the Boolean poset $(2^{[n]}, \subseteq)$ has a least upper bound and a greatest lower bound.

Problem 7

Prove or disprove: If a poset $(P, \leq)$ has a single minimal element, then it is a least element as well.

Problem 8

Let $n \geq 1$ and $C = (\{+, 0, -\}, \leq)$, where $\leq$ has the two covers $+ \leq 0$ and $- \leq 0$. Let C^n be the n-th power of C.

a) What is the rank of C^n? What is the number of minimal elements?

b) What is the number of elements of rank $i \geq 0$ in C^n? (Do the sequences sound familiar?)

c) Show that this is a join-semilattice.

Problem 9

Give an example of lattice L for which the distributivity axiom **is not true**:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \qquad \text{for all } x, y, z \in L.$$

Problem 10

Prove that a finite lattice always has a greatest element and a least element.