Exercise Sheet 6 (Partial Solutions)

Discrete Mathematics I - SoSe17

Lecturer Jean-Philippe Labbé

Tutors Johanna Steinmeyer and Patrick Morris

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You should try to solve all of the exercises below, and select three to four solutions to be submitted and graded. We encourage you to submit in pairs, please remember to indicate the author of each individual solution and the name of both team members on the cover sheet.

Problem 1

Let $A_{n+5} - 5A_{n+4} + 9A_{n+3} - 9A_{n+2} + 8A_{n+1} - 4A_n = 0$.

a) Find the matrix M such that $MX_0 = X_1$, where $X_i = (A_i, A_{i+1}, \ldots, A_{i+4})^t$ and give its characteristic polynomial.

b) Give the eigenvalues and a basis for the associated *generalized* eigenspaces of M. In other words, give the Jordan form and the associated transition matrix T, whose columns are generalized eigenvectors.

c) Give the value $M^{10}C$, for all columns C of the transition matrix T. Get a general formula for M^nT.

Problem 2

Provide bijections between the following four sets, so as to show that they are all equinumerous.

a) Let T_n be the set of triangulations of a convex $(n+2)$-gon.

b) Let P_n be the set of parenthesation of $n+1$ factors in a non-associative binary operation. For $n = 2$, there are two ways: $a(bc)$ and $(ab)c$.

c) Let D_n be the set of lattice paths from $(0,0)$ to (n,n) that stay above the diagonal.

d) Let S_n be the set of strings of n 1's and n -1's such that every initial partial sum is nonnegative.

Problem 3

We subdivide a group of people into subgroups A, B, and C, and ask them to form a line. We also require that A have an odd number of people, B have an even number of people. How many ways are there to do this?

Let n be the number of people in the group. Yes, people are distinct. The subgroups are labeled, so for a fixed person, it matters in which group it is attributed. First, let $\mathcal{A}^e(x), \mathcal{B}^e(x)$, and $\mathcal{C}^e(x)$ be the generating functions for the subgroups A, B, and C, respectively. We have

$$\mathcal{A}^e(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \frac{e^x - e^{-x}}{2},$$

since there is one way to create a set of odd cardinality. Similarly,

$$\mathcal{B}^e(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \cdots = \frac{e^x + e^{-x}}{2},$$

assuming that B could be empty. Finally,

$$\mathcal{C}^e(x) = e^x.$$

In order to form the three subgroups, we multiply the exponential generating functions:

$$\mathcal{A}^e(x)\mathcal{B}^e(x)\mathcal{C}^e(x) = \frac{e^{3x} - e^{-x}}{4}.$$

The coefficient of x^n in this exponential generating function is $\frac{3^n+(-1)^{n+1}}{4}$. This number gives the number of ways to form the subgroups, but not to order them on a line. For example, when $n = 2$, the first person can be in A and the second in C or the first person can be in C and the second in A. These are the 2 ways to form the subgroups. Then, we multiply by $n!$ to account for the number of way to put these people on a line at the end. So the answer is $n! \cdot \frac{3^n+(-1)^{n+1}}{4}$.

Problem 4

In individual carries a certain number of coins. Suppose it consists of at most 5 five cents, 4 ten cents, 3 twenty cents, 3 fifty cents and 2 one euro. How many ways can this person make exact change for two euros?

Problem 5

Show that in a country that has 1-cent, 2-cent, and 3-cent coins only, the number of ways of changing n cents is exactly the integer nearest to $(n+3)^2/12$.

Problem 6

The generating function of the Fibonacci numbers is $\mathcal{F}(x) = \frac{x}{1-x-x^2}$. From this, determine the generating function of the Fibonacci numbers with even index $(F_{2n})_{n\geq 0}$.

Problem 7

Solve the recurrence $A_0 = A_1 = 1$, $A_n = A_{n-1} + (n-1)A_{n-2}$ for $n \geq 2$, using exponential generating functions.

Problem 8

Let H_n denote the number of n-digit numbers with digits 1, 2 or 3 where the number of 1's is even, the number of 2's is at least three, and the number of 3's is at most four. Determine the exponential generating function $\mathcal{H}^{(e)}(x)$ for the sequence $(H_n)_{n\geq 0}$.

The exponential generating function of the 1's is $\frac{e^x - e^{-x}}{2}$, the one for the 2's is $e^x - (1 + x + \frac{x^2}{2})$ and the one for 4's is $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$. The multiplication of these three exp. generating function gives $\mathcal{H}^{(e)}(x)$.

Problem 9

Formulate a combinatorial problem that leads to the following generating function.

$$(1 + x + x^2)(1 + x^2 + x^4 + x^6)(1 + x^2 + x^4 + \cdots)(x + x^2 + x^3 + \cdots)$$

Problem 10

There are $2n$ people in line to get into a club. Admission is 5 euros. Of the $2n$ people, n have a 5 euro bill and n have a 10 euro bill. The admission of the club foolishly start with an empty cash register. In how many ways can the people line up so that whenever a person with a 10 euro bill buys an entry, the admission has a 5 euro bill to make the change?