Problem 1 (sophia elia)

Let $A_{n+5} - 5A_{n+4} + 9A_{n+3} - 9A_{n+2} + 8A_{n+1} - 4A_n = 0$

ⓐ Find the matrix M such that $MX_0 = X_1$, where $x_i = (A_i, A_{i+1}, \dots, A_{i+4})$

and give its characteristic polynomial

Solving for A_{n+5} immediately tells us the last row of M:

$$A_{n+5} = 5A_{n+4} - 9A_{n+3} + 9A_{n+2} - 8A_{n+1} + 4A_n$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 4 & -8 & 9 & -9 & 5 \end{bmatrix}}_{M} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{bmatrix} \checkmark$$

To find M's characteristic polynomial, we calculate the determinant of $M - \lambda$:

$$\det(M - \lambda I) = +4(1) + 8\left(\det\begin{bmatrix} -\lambda & & & \\ & 1 & & \\ & -\lambda & 1 & \\ & & -\lambda & 1 \end{bmatrix}\right) - 9\left(\det\begin{bmatrix} -\lambda & 1 & & \\ & -\lambda & & \\ & & 1 & \\ & & -\lambda & 1 \end{bmatrix}\right) + 9\left(\det\begin{bmatrix} -\lambda & 1 & & \\ & -\lambda & 1 & \\ & & -\lambda & \\ & & & 1 \end{bmatrix}\right) + 5(-\lambda)^4 + (-\lambda$$

$\Rightarrow$ the char poly is

$$\boxed{0 = \lambda^5 - 5\lambda^4 + 9\lambda^3 - 9\lambda^2 + 8\lambda - 4} \checkmark$$

ⓑ give the eigenvalues and a basis for the associated generalized eigenspaces of M.

1 is a root of the characteristic polynomial:

$$\lambda^5 - 5\lambda^4 + 9\lambda^3 - 9\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda^4 - 4\lambda^3 + 5\lambda^2 - 4\lambda + 4)$$

2 is a root of the second factor:

$$= (\lambda - 1)(\lambda - 2)(\lambda^3 - 2\lambda^2 + \lambda - 2)$$

2 is a root of the third factor

$$= (\lambda - 1)(\lambda - 2)^2(\lambda^2 + 1)$$

$$= (\lambda - 1)(\lambda - 2)^2(\lambda - i)(\lambda + i)$$

The eigenvalues are 1, 2 - with multiplicity 2, i, and $-i$.

D continued.

First I calculate the ordinary eigenvectors. An eigenvector $\vec{v}$ satisfies the equation $A\vec{v} = \lambda\vec{v}$. So $(A-\lambda I)\vec{v} = 0$.

$(\lambda = 1)$. $M - I = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 4 & -8 & 9 & -9 & 4 \end{bmatrix}$. Then I must solve

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 4 & -8 & 9 & -9 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = 0$$

I have the following five equations: $-v_1 + v_2 = 0$ (1)

$\Rightarrow v_2 = v_1$

$-v_2 + v_3 = 0$ (2)

$\Rightarrow v_3 = v_2 = v_1$

$-v_3 + v_4 = 0$ (3)

$\Rightarrow v_4 = v_3 = v_2 = v_1$

$-v_4 + v_5 = 0$ (4)

$\Rightarrow v_5 = v_4 = v_3 = v_2 = v_1$

$$4v_1 - 8v_2 + 9v_3 - 9v_4 + 4v_5 = 0$$

$$4v_1 - 8v_1 + 9v_1 - 9v_1 + 4v_1 = 0$$ substituting what we know

$$0 v_1 = 0$$

So we can choose any value for v_1

Choose 1.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Problem 1 b continued

$\lambda = 2$

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 4 & -8 & 9 & -9 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = 0$$

yields the following equations

$\Rightarrow \quad -2v_1 + v_2 = 0$

$\Rightarrow v_2 = 2v_1$

likewise we have

$v_{i+1} = 2v_i$

$4v_1 - 8v_2 + 9v_3 - 9v_4 + 3v_5 = 0$

substituting in v_1:

$$4v_1 - 8(2v_1) + 9(4v_1) - 9(8v_1) + 3(16v_1) = 0$$

$$v_1(4 - 16 + 36 - 72 + 48) = 0$$

so v_1 is free. we can choose

$$\vec{V}_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{bmatrix}$$

$\lambda = i$

$$\begin{bmatrix} -i & 1 & 0 & 0 & 0 \\ 0 & -i & 1 & 0 & 0 \\ 0 & 0 & -i & 1 & 0 \\ 0 & 0 & 0 & -i & 1 \\ 4 & -8 & 9 & -9 & 5-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = 0$$

gives the following equations

$V_2 = i \cdot v_1$

$V_{j+1} = i \cdot v_j$

$$4v_1 - 8v_2 + 9v_3 - 9v_4 + 5v_5 - iv_5 = 0$$

$$\Rightarrow 4v_1 - 8iv_1 - 9v_1 + 9iv_1 + 5v_1 - iv_1$$

v_1 is free

$$\vec{V}_i = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \\ 1 \end{bmatrix}$$

$\lambda = -i$ we'll get the same equations we just multiply by $-i$ each time

$$\vec{V}_{-i} = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \\ 1 \end{bmatrix}$$

1b continued

We compute the generalized eigenvector corresponding to $\lambda = 2$ by solving $(M-2I)V_{2'} = V_2$

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 4 & -8 & 9 & -9 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{bmatrix}$$

yields:

$-2v_1 + v_2 = 1 \Rightarrow v_2 = 1 + 2v_1$

$-2v_2 + v_3 = 2 \Rightarrow v_3 = 2 + 2v_2 \Rightarrow v_3 = 2 + 2 + 4v_1 = 4 + 4v_1$

$-2v_3 + v_4 = 4 \Rightarrow v_4 = 4 + 2v_3 \Rightarrow v_4 = 4 + 8 + 8v_1 = 12 + 8v_1$

$-2v_4 + v_5 = 8 \Rightarrow v_5 = 8 + 2v_4 \Rightarrow v_5 = 8 + 24 + 16v_1 = 32 + 16v_1$

$+4v_1 - 8v_2 + 9v_3 - 9v_4 + 3v_5 = 16$

plugging in:

$$\cancel{V_{2'} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 20 \\ 48 \end{bmatrix}} \Rightarrow$$

$+4v_1 - 8(1+2v_1) + 9(4+4v_1) - 9(12+8v_1) + 3(32+16v_1) = 16$

$+4v_1 - 8 - 16v_1 + 36 + 36v_1 - 108 - 72v_1 + 96 + 48v_1 = 16$

$\Rightarrow v_1$ is free

$$V_{2'} = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 12 \\ 32 \end{bmatrix}$$

This concludes the search.

problem 1c. Give the value $M^{10}C$, for all columns C of the transition matrix T. Get a general formula for M^nT.

My transition matrix T is made up of my generalized eigenvectors

$$T = \begin{array}{c} \begin{matrix} \vec{v}_1 & \vec{v}_i & \vec{v}_{-i} & \vec{v}_2 & \vec{v}_{2'} \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & i & -i & 2 & 1 \\ 1 & -1 & -1 & 4 & 4 \\ 1 & -i & i & 8 & 12 \\ 1 & 1 & 1 & 16 & 32 \end{bmatrix} \end{array}$$

For the first four columns of T, I have regular eigenvectors which satisfy $M\vec{v} = \lambda v \Rightarrow M^{10}\vec{v} = \lambda^{10} v$, or more generally $M^p\vec{v} = \lambda^p v$.

Explicitly $M^{10}(v_1) = v_1$

$M^{10}(v_i) = -v_i$

$M^{10}(v_{-i}) = -v_{-i}$

$M^{10}(v_2) = 2^{10} v_2 = 1024 v_2$

problem 1 c continued.

To calculate the generalized eigenvector v_2' we used

$$M v_2' - 2 I v_2' = v_2$$

$$\Rightarrow \quad M v_2' = v_2 + 2 v_2'$$

$$\text{so} \quad M^p v_2' = M^{p-1} \cdot M \cdot v_2' = M^{p-1} \cdot (v_2 + 2v_2')$$

$$= 2M^{p-1} v_2' + \cancel{2} M^{p-1} v_2$$

$$= M^{p-2} v_2' + 2M^{p-2} v_2' + 2^{p-1} v_2$$

$$= v_2' + p \cdot 2^p v_2$$

(in particular $M^{10} v_2' = v_2' + 10 \cdot 2^{10} v_2$)

should be $2^p v_2' + p 2^{p-1} v_2$

get a general formula for $M^n T$.

$$\begin{bmatrix} & & \\ & M^n & \\ & & \end{bmatrix} \begin{bmatrix} & & & & \\ v_1 & v_i & v_{-i} & v_2 & v_{2'} \\ & & & & \end{bmatrix}$$

$$= \begin{bmatrix} & & & & \\ v_1 & (i)^n v_i & (-i)^n v_{-i} & (2)^n v_2 & 2^n v_{2'} + n \cdot 2^n v_2 \\ & & & & \end{bmatrix}$$

✓ Little slip up, but all the right ideas!

$\{a,b\} \sim \{c,d\}$ ✓

Problem 3

Since the people are clearly distinct, we need to use exponential generating functions $A^{(e)}(x)$, $B^{(e)}(x)$ and $C^{(e)}(x)$ respectively. These are all variants of the exponential function, with the simplest being that for C, where there are no restrictions.

$$C^{(e)}(x) = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n\geq 0} \frac{x^n}{n!} = e^x$$

For A, we must have only the odd terms and therefore we take out the even terms:

$$A^{(e)}(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!} = \frac{e^x - e^{-x}}{2}$$

Similarly for B, we remove the odd terms to leave only even terms:

$$B^{(e)}(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{n\geq 0} \frac{x^{2n}}{(2n)!} = \frac{e^x + e^{-x}}{2}$$

To find the number of ways to form subgroups A, B and C, look at the product of these three.

$$A^{(e)}(x)B^{(e)}(x)C^{(e)}(x) = \frac{1}{2}\left(e^x - e^{-x}\right)\frac{1}{2}\left(e^x + e^{-x}\right)e^x = \frac{1}{4}\left(e^{3x} - e^{-x}\right)$$
$$= \frac{1}{4}\left(\sum_{n\geq 0} \frac{(3x)^n}{n!} - \sum_{n\geq 0} \frac{(-x)^n}{n!}\right) = \sum_{n\geq 0} \frac{3^n - (-1)^n}{4} \frac{x^n}{n!}$$

3

Giving $\left(3^n - (-1)^n\right)/4$ ways to make the subgroups.
Finally the number of ways of lining up the whole group will be $n!$ and as this is independent of choosing the subgroups by the multiplication principle we have the number of ways to form subgroups and then to form a line is

$$\frac{\left(3^n - (-1)^n\right)n!}{4}$$

✓ Great

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Problem 6 - Author: Christina

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and observe that $\sum_{n=0}^{\infty} a_{2n} x^{2n} = \frac{A(x)+A(-x)}{2}$.

Now, since $F(x) = \frac{x}{1-x-x^2}$, we conclude that the generating function of the Fibonacci numbers with even index is:

$$F_E(x) = \sum_{n=0}^{\infty} F_{2n} x^n = \frac{F\left(x^{1/2}\right) + F(-x^{1/2})}{2} = \frac{x}{1-3x+x^2}$$

3

Problem 5 - Author: Christina

Problem 8

As exponential generating functions take ordered choice into account the exponential generating function for the desired amount of $n-$digit number can be, analogously to the coin change example, determined by:

$$\begin{aligned}\mathcal{H}^{(e)}(x) &= \left(\sum_{i=0}^{\infty} \frac{x^{2i}}{2i!}\right)\left(\sum_{i=3}^{\infty} \frac{x^{i}}{i!}\right)\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}\right) \\ &= \cosh(x)\left(e^x-1-x-\frac{x^2}{2}\right)\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}\right)\end{aligned}$$

✓ 3

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