Exercise Sheet 5 (Partial Solutions)

Discrete Mathematics I - SoSe17

Lecturer Jean-Philippe Labbé

Tutors Johanna Steinmeyer and Patrick Morris

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You should try to solve all of the exercises below, and select three to four solutions to be submitted and graded. We encourage you to submit in pairs, please remember to indicate the author of each individual solution and the name of both team members on the cover sheet.

Problem 1

Which weak partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of n maximizes the number of permutations of type λ?

Problem 2

Solve the following recurrence relations.

a) $A_0 = a$, $A_n = A_{n-1} + n$.

b) $A_0 = a$, $A_n = bA_{n-1} - c$.

c) $A_0 = 2$, $A_n = (A_{n-1})^n$.

d) $A_0 = 1$, $A_n = 2^n - A_{n-1}$.

e) $A_0 = 0$, $A_n = \frac{n}{n+1} A_{n-1} + 1$.

f) $A_0 = 5$, $A_n = nA_{n-1}$.

b) If $b = 1$, then, by exhaustion, $A_n = a - nc$. Otherwise, the characteristic polynomial is $\chi(A) = (x - b)$ and $g(n) = -c$ is a polynomial of degree 0. Hence, since $b \neq 1$, the general solution of the recurrence relation is $\alpha b^n + \beta$ for $\alpha, \beta \in \mathbb{R}$. Solving for α and β, one gets $\alpha = a - \frac{c}{b-1}$ and $\beta = \frac{c}{b-1}$. Thus the solution is $A_n = ab^n + \frac{c}{b-1}(1 - b^n)$.

Problem 3

Find and solve the recurrence relation for the number A_n of strings of $0, 1$, or 2 of length n,

a) that do not contain the subsequence 00, nor 11.

b) that contain the subsequence 00 or 11.

c) that contain an even number of zeros.

c) Let O_n denote the number of strings with an odd number of zeros and let E_n denote the number of strings with an even number of zeros. Clearly $3^n = E_n + O_n$. Any string of length n with an even number of zeros finishes with either a 0, a 1 or a 2. Hence if it finishes with a 1 or a 2, we can remove the last entry to get a string of length $n-1$ that has an even number of zeros, hence there are $2E_{n-1}$ such strings. Finally, we have to add the strings ending with 0. Since removing the zero we get strings with an odd number of zeros, we have that there are $3^{n-1} - E_{n-1}$ such strings. Hence $E_n = 2E_{n-1} + 3^{n-1} - E_{n-1} = E_{n-1} + 3^{n-1}$. We have $E_2 = 1$ and $E_3 = 6$. Hence the general solution is of the form $\alpha + \beta 3^n$. Solving for α and β, we get $\alpha = -\frac{3}{2}$, $\beta = \frac{5}{18}$. Hence $E_n = -\frac{3}{2} + \frac{5\cdot 3^{n-2}}{2} = \frac{5\cdot 3^{n-2}-3}{2}$ for $n \geq 2$.

Problem 4

In how many ways can a $2\times n$ grid be filled with dominos of size 1×2 and 2×2?

Problem 5

Consider the recurrence relation

$$A_n = A_{n-1} + A_{n-2} - A_{n-3} + (-1)^n + 2n, \quad n \geq 3. \qquad (R)$$

a) Find the general solution to the recurrence relation.

b) Show that the general solution you obtained is a general solution, i.e., show that for any choice of parameters, it give a particular solution, and any given particular solution is obtained from the general solution for some choice of parameters.

c) Find the initial conditions to be imposed so that the sequence $(A_n)_{n\in\mathbb{N}}$ is a solution to (R) and of

$$A_n = 8A_{n-1} + 5A_{n-2} + n(-1)^n - 65n - 2n^3, \quad n \geq 2.$$

a) The characteristic polynomial of the recurrence relation is $\chi(A) = x^3 - x^2 - x + 1 = (x+1)(x-1)^2$. Hence the general solution of the homogeneous recurrence relation is $\alpha(-1)^n + \gamma + \delta n$. Since -1 is a root, the particular solution to add (using the superposition principle) for the part $(-1)^n$ in the function $g(n)$ is $\beta n(-1)^n$. Further, we have a polynomial of degree 1 in $g(n)$ and 1 is a root of multiplicity 2, so we add $\epsilon n^2 + \phi n^3$. Therefore the general solution to the non-homogeneous recurrence relation is

$$(R) \qquad (-1)^n(\alpha + \beta n) + (\gamma + \delta n + \epsilon n^2 + \phi n^3)$$

c) We plug in the general solution into the second recurrence relation:

$$\begin{aligned}(-1)^n(\alpha+\beta n) + (\gamma + \delta n + \epsilon n^2 + \phi n^3) =& 8[(-1)^{n-1}(\alpha + \beta(n-1)) + (\gamma + \delta(n-1) + \epsilon(n-1)^2 + \phi(n-1)^3)] \\ &+ 5[(-1)^{n-2}(\alpha + \beta(n-2)) + (\gamma + \delta(n-2) + \epsilon(n-2)^2 + \phi(n-2)^3)] \\ &+ n(-1)^n - 65n - 2n^3\end{aligned}$$

We deduce that $(-1)^n(\alpha + \beta n) = (-1)^n[(1-3\beta)n - 3\alpha - 2\beta]$, so we deduce that $\beta = 1/4$ and $\alpha = -1/8$. Similarly for the polynomial, we deduce that $\phi = \frac{1}{6}, \epsilon = \frac{3}{4}, \delta = \frac{13}{2}, \gamma = \frac{26}{3}$. Plugging these values in the general solution for (R) above and evaluating for $n = 0, 1, 2$, we get $A_0 = \frac{205}{24}, A_1 = \frac{383}{24}$, and $A_2 = \frac{211}{8}$.

Problem 6

Solve the following recurrence relations

a) $A_1 = A_0 = 0,\ A_n = n^2 + n + 2\sum_{i=0}^{n-2} A_i$.

b) $A_0 = 1 \quad B_0 = 2,\ A_n = 3A_{n-1} + 2B_{n-1} + 3^n$ and $B_n = A_{n-1} + 2B_{n-1}$.

a) By shifting the recurrence relation:

$$A_{n+1} = (n+1)^2 + (n+1) + 2\sum_{i=0}^{n-1} A_i$$

$$A_n = n^2 + n + 2\sum_{i=0}^{n-2} A_i$$

Taking the difference yields

$$A_{n+1} - A_n = 2(n+1) + 2A_{n-1}.$$

Shifting again and substracting:

$$A_{n+2} - 2A_{n+1} - A_n + 2A_{n-1} = 2.$$

Equating with the shift gives:

$$A_{n+3} - 3A_{n+2} + A_{n+1} + 3A_n - 2A_{n-1} = 0.$$

We compute $A(2) = 6$ and $A(3) = 12$. The characteristic polynomial is $\chi(A) = x^4 - 3x^3 + x^2 + 3x - 2 = (x-2)(x+1)(x-1)^2$. Hence the general solution is $A(n) = \alpha 2^n + \beta(-1)^n + \gamma n + \delta$. Finally, one solves it for α, β, γ and δ. We get $\alpha = 4, \beta = 2, \gamma = -6$, and $\delta = 0$. Thus $A_n = 2^{n+2} + 2(-1)^n - 6n$.

Problem 7

Show that

$$f(x) = \frac{1}{(1-x)^3} = \sum_{i=0}^{\infty} \frac{(i+1)(i+2)}{2} x^i.$$

Hint: Interpret the LHS using the Multiplication Principle and use some result seen in class that counts the objects given by this interpretation.

Problem 8

Use basic generating functions and operations on them to obtain how many ways there are to make change for a hundred euro bill using five, ten twenty, fifty and hundred dollar bills.

Problem 9

Find the generating functions for the following sequences (in closed form):

a) $(0, 0, 0, 0, -6, 6, -6, 6, -6, \dots)$.

b) $(1, 0, 1, 0, 1, 0, \dots)$.

c) $(1, 2, 1, 4, 1, 8, \dots)$.

a) $-6x^4 + 6x^5 - 6x^6 + 6x^7 \dots = -6x^4(1 - x + x^2 - x^3 + \dots) = \frac{-6x^4}{1+x}$.

b) $1 + x^2 + x^4 + x^6 + \dots = \frac{1}{1-x^2}$.

c) $1 + 2x + 1x^2 + 4x^3 + 1x^4 + 8x^5 + \dots = (1 + x^2 + x^4 + x^6 + \dots) + 2x(1 + 2x^2 + (2x^2)^2 + (2x^2)^3) = \frac{1}{1-x^2} + \frac{2x}{1-2x^2}$.

Problem 10

a) Give the generating function for the sequence $(D_n)_{n\in\mathbb{N}}$, of partitions of n using distinct parts.

b) Give the generating function for the sequence $(O_n)_{n\in\mathbb{N}}$, of partitions of n using only odd parts.

c) Prove that $D_n = O_n$, for all n.