Exercise Sheet 5

Discrete Mathematics I - SoSe17

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You should try to solve all of the exercises below, and select three to four solutions to be submitted and graded. We encourage you to submit in pairs, please remember to indicate the author of each individual solution and the name of both team members on the cover sheet.

Problem 1

Which weak partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of n maximizes the number of permutations of type λ?

Problem 2

Solve the following recurrence relations.

a) $A_0 = a$, $A_n = A_{n-1} + n$.

b) $A_0 = a$, $A_n = bA_{n-1} - c$.

c) $A_0 = 2$, $A_n = (A_{n-1})^n$.

d) $A_0 = 1$, $A_n = 2^n - A_{n-1}$.

e) $A_0 = 0$, $A_n = \frac{n}{n+1}A_{n-1} + 1$.

f) $A_0 = 5$, $A_n = nA_{n-1}$.

Problem 3

Find and solve the recurrence relation for the number A_n of strings of $0, 1$, or 2 of length n,

a) that do not contain the subsequence 00, nor 11.

b) that contain the subsequence 00 or 11.

c) that contain an even number of zeros.

Problem 4

In how many ways can a $2 \times n$ grid be filled with dominos of size 1×2 and 2×2?

Problem 5

Consider the recurrence relation

$$A_n = A_{n-1} + A_{n-2} - A_{n-3} + (-1)^n + 2n, \quad n \geq 3. \qquad (R)$$

a) Find the general solution to the recurrence relation.

b) Show that the general solution you obtained is a general solution, i.e., show that for any choice of parameters, it give a particular solution, and any given particular solution is obtained from the general solution for some choice of parameters.

c) Find the initial conditions to be imposed so that the sequence $(A_n)_{n\in\mathbb{N}}$ is a solution to (R) and of

$$A_n = 8A_{n-1} + 5A_{n-2} + n(-1)^n - 65n - 2n^3, \quad n \geq 2.$$

Problem 6

Solve the following recurrence relations

a) $A_1 = A_0 = 0$, $A_n = n^2 + n + 2\sum_{i=0}^{n-2} A_i$.

b) $A_0 = 1 \quad B_0 = 2$, $A_n = 3A_{n-1} + 2B_{n-1} + 3^n$ and $B_n = A_{n-1} + 2B_{n-1}$.

Problem 7

Show that

$$f(x) = \frac{1}{(1-x)^3} = \sum_{i=0}^{\infty} \frac{(i+1)(i+2)}{2} x^i.$$

Hint: Interpret the LHS using the Multiplication Principle and use some result seen in class that counts the objects given by this interpretation.

Problem 8

Use basic generating functions and operations on them to obtain how many ways there are to make change for a hundred euro bill using five, ten twenty, fifty and hundred dollar bills.

Problem 9

Find the generating functions for the following sequences (in closed form):

a) $(0, 0, 0, 0, -6, 6, -6, 6, -6, \dots)$.

b) $(1, 0, 1, 0, 1, 0, \dots)$.

c) $(1, 2, 1, 4, 1, 8, \dots)$.

Problem 10

a) Give the generating function for the sequence $(D_n)_{n\in\mathbb{N}}$, of partitions of n using distinct parts.

b) Give the generating function for the sequence $(O_n)_{n\in\mathbb{N}}$, of partitions of n using only odd parts.

c) Prove that $D_n = O_n$, for all n.