Exercise Sheet 4 (Partial Solutions)

Discrete Mathematics I - SoSe17

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You should try to solve all of the exercises below, and select three to four solutions to be submitted and graded. We encourage you to submit in pairs, please remember to indicate the author of each individual solution and the name of both team members on the cover sheet.

Problem 1

Find the number of lattice paths from $(0,0,0,0)$ to $(7,5,3,9)$ that pass through $(5,3,1,5)$ but not $(4,2,0,3)$.

Let C be the set of paths from $(0,0,0,0)$ to $(7,5,3,9)$. Let $A \subset C$ be the set of paths in C which pass through $(5,3,1,5)$. Let $B \subset C$ be the set of paths in C which pass through $(4,2,0,3)$. We look for $A \setminus B = A \setminus (A \cap B)$. We use multinomial coefficients to count the cardinalities of A, and $A \cap B$. First, the number of paths from $(0,0,0,0)$ to $(7,5,3,9)$ passing through $(5,3,1,5)$ is $\binom{14}{5,3,1,5}\binom{10}{2,2,2,4}$ by the multiplication principle. Second, the number of paths that pass through both $(4,2,0,3)$ and $(5,3,1,5)$ is $\binom{9}{4,2,3}\binom{5}{1,1,1,2}\binom{10}{2,2,2,4}$. Therefore the number of paths passing through $(5,3,1,5)$ but not $(4,2,0,3)$ is

$$\binom{14}{5,3,1,5}\binom{10}{2,2,2,4} - \binom{9}{4,2,3}\binom{5}{1,1,1,2}\binom{10}{2,2,2,4}.$$

Problem 2

Prove the following identities:

a) $\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_1,\lambda_2,\lambda_3\geq 0}} \binom{n}{\lambda_1,\lambda_2,\lambda_3} = 3^n$,

b) $\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_1,\lambda_2,\lambda_3\geq 0}} \binom{n}{\lambda_1,\lambda_2,\lambda_3}\lambda_1 = n3^{n-1}$,

c) $\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_1,\lambda_2,\lambda_3\geq 0}} \binom{n}{\lambda_1,\lambda_2,\lambda_3}\lambda_1\lambda_2 = n(n-1)3^{n-2}$.

Problem 3

Let D_n be the number of dérangements of $[n]$. Show that the sequence $(D_n)_{n\in\mathbb{N}}$ satisfies the recurrence relation

$$D_n = n \cdot D_{n-1} + (-1)^n,$$

with initial condition $D_0 = 1$.

J.B. Remmel. A Note on a Recursion for the Number of Derangements, Europ. J. Combinatorics (1983) 4, 371-374.

Problem 4

A triangulation of a regular n-gon is a way of placing $n-3$ non-intersecting diagonals in the n-gon. Give a recurrence with appropriate initial conditions for T_n, the number of triangulations on the regular n-gon. Identify the recurrence as completely as possible.

Problem 5

Find a closed form for the recurrence relations on $(R_n)_{n\in\mathbb{N}}$

a) $R_n = R_{n-1} + 6R_{n-2}$, where $R_0 = 0$ and $R_1 = 1$,

b) $R_n = 5R_{n-1} + 29R_{n-2} - 105R_{n-3}$, where $R_0 = 0$, $R_1 = 1$ and $R_2 = 1$,

c) $R_n = 8R_{n-1} - 21R_{n-2} + 18R_{n-3}$, where $R_0 = R_1 = 0$, and $R_2 = 1$.

b) The characteristic polynomial is $\chi(R) = x^3 - 5x^2 - 29x + 105 = (x+5)(x-3)(x-7)$. Therefore, the general solution of the homogeneous linear recurrence relation is $\alpha(-5)^n + \beta 3^n + \gamma 7^n$. Now we solve

$$\begin{array}{rcl|l} \alpha+\beta+\gamma & = & 0 & R_0 \\ -5\alpha+3\beta+7\gamma & = & 1 & R_1 \\ 25\alpha+9\beta+49\gamma & = & 1 & R_2 \end{array}$$

and obtain: $\alpha = \frac{-3}{32}, \beta = \frac{1}{32}, \gamma = \frac{1}{16}$. Therefore, the solution is $R_n = \frac{-3}{32}(-5)^n + \frac{3^n}{32} + \frac{7^n}{16}$, for $n \geq 0$.

Problem 6

Find the general solution for the recurrence

$$R_{n+5} - 9R_{n+4} + 31R_{n+3} - 63R_{n+2} + 108R_{n+1} - 108R_n = 0.$$

Problem 7

Show that among any 5 points on a plane, no three points through a line, 4 of them must form a convex quadrangle.

Let $x_1, \ldots, x_5 \in \mathbb{R}^2$. If the convex hull of x_1 to x_5 is a pentagon or a quadrangle, the statement is obvious. Furthermore, the convex hull can not be contained in a line by assumption. Hence, we are left with the case where 3 points are in convex position, say x_1, x_2 and x_3, and the other two points lie strictly inside the triangle spanned by them.

Pick one of the interior points, say x_4. Again since no three points lie on a line, the fifth point has to be in the interior of the three triangles determined by the sides of the bigger triangle and the point x_4. Therefore, the line spanned by x_4 and x_5 intersect two sides of the bigger triangle, say the two touching x_1. Then, x_2 to x_5 form a convex quadrangle.

Problem 8

Show, using a combinatorial argument that

$$\sum_{k=0}^{n} \binom{n}{k} D_k = n!,$$

where D_k is the number of dérangements of $[k]$.

Problem 9

Say if the given sequences are particular solutions of the recurrence relation $A_n = 8A_{n-1} - 16A_{n-2}$.

a) $(0)_{n\in\mathbb{N}}$,
b) $(2^n)_{n\in\mathbb{N}}$,
c) $(4^n)_{n\in\mathbb{N}}$,
d) $(n2^{2n})_{n\in\mathbb{N}}$,
e) $(2 \cdot 4^n + 3n4^n)_{n\in\mathbb{N}}$,
f) $(n^2 4^n)_{n\in\mathbb{N}}$,

The characteristic polynomial of the recurrence relation is $\chi(A) = x^2 - 8x + 16 = (x-4)^2$. Hence, any particular solution is of the form $\alpha 4^n + \beta n 4^n$, for some $\alpha, \beta \in \mathbb{R}$.

a) Yes, $\alpha = \beta = 0$,

b) No, the system $2^n = \alpha 4^n + \beta n 4^n$, has no solution valid for every n,

c) Yes, $\alpha = 1, \beta = 0$,

d) Yes, $\alpha = 0, \beta = 1$,

e) Yes, $\alpha = 2, \beta = 3$,

f) No, the system $n^2 4^n = \alpha 4^n + \beta n 4^n$ has no solution valid for every n.

Problem 10

A worker was hired in 1995 with 3 000Euros/month. Every year the worker receives an increase of 100Euros plus 2% of the monthly salary of the previous year.

a) Give a recurrence relation to compute the monthly salary at year $1995 + n$.
b) Solve this recurrence relation.
c) Compute the monthly salary in 2017.