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Exercise Sheet 4. Niloofar ③

problem 2: prove the following identities.

a) $$\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_1,\lambda_2,\lambda_3 \geq 0}} \binom{n}{\lambda_1,\lambda_2,\lambda_3} = 3^n$$

Combinatorial prove: we have n balls (labelled!). we want to paint them in 3 colors. $\lambda_1 :=$ # red balls, $\lambda_2 =$ # blue balls

$\lambda_3 :=$ # green balls

LHS: counts the number of possible ways that we divide our balls into 3 groups $\Lambda_1, \Lambda_2, \Lambda_3$ where $|\Lambda_1| = \lambda_1$, $|\Lambda_2| = \lambda_2$, $|\Lambda_3| = \lambda_3$ then paint them in red, blue, green the number of ways to do it when we know the number of balls in each group is equal to $\binom{n}{\lambda_1\ \lambda_2\ \lambda_3}$ but here as we just know $\lambda_1+\lambda_2+\lambda_3 = n$ by A.P. we sum up $\binom{n}{\lambda_1,\lambda_2,\lambda_3}$ for different $\lambda_1, \lambda_2, \lambda_3 \geq 0$ $\Rightarrow$

LHS is $\displaystyle\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n\\ \lambda_i \geq 0}} \binom{n}{\lambda_1\ \lambda_2\ \lambda_3}$

RHS: we have n balls, and for each ball we have 3 option for coloring by M.P:

$$\underbrace{\underbrace{\begin{matrix}\text{red}\\ \text{blue}\\ \text{green}\end{matrix}}_{3}\ \underbrace{\begin{matrix}\text{red}\\ \text{blue}\\ \text{green}\end{matrix}}_{3}\ \underbrace{\begin{matrix}\text{red}\\ \text{blue}\\ \text{green}\end{matrix}}_{3}\ \cdots\ \underbrace{\begin{matrix}\text{red}\\ \text{blue}\\ \text{green}\end{matrix}}_{3}}_{n \text{ balls}} = 3^n \checkmark$$

Nice!

b) $\displaystyle\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n\\ \lambda_1,\lambda_2,\lambda_3 \geq 0}} \lambda_1 \binom{n}{\lambda_1\ \lambda_2\ \lambda_3} = n\,3^{n-1}$

Like first part, we want to paint n balls into $\underline{3}$ colors but here; we have from specific color at least $\underline{1}$ ball:

LHS: $\displaystyle\sum_{\substack{\lambda_1+\lambda_2+\lambda_3\\ \lambda_i \geq 0}} \lambda_1 \binom{n}{\lambda_1\ \lambda_2\ \lambda_3} = \cancelto{0}{\sum_{\substack{\lambda_2+\lambda_3=n\\ \lambda_1=0\\ \lambda_2,\lambda_3\geq 0}} \lambda_1 \binom{n}{\lambda_1\ \lambda_2\ \lambda_3}}$

$\displaystyle + \sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n\\ \lambda_1 \geq 1\\ \lambda_2,\lambda_3 \geq 0}} \binom{n}{\lambda_1\ \lambda_2\ \lambda_3} \lambda_1$

in LHS we first divide balls into 3 coloring groups

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rest of 2:

red, blue, green like (a) and amoung red balls we want to choose one of them and stack ($*$) on it if we know the number of each groups, the number of possible ways is by M.P. $\binom{n}{\lambda_1 \lambda_2 \lambda_3} \cdot \lambda_1$

possible way to stack star on one of red balls!

then we have to sum it up to get the number of ways for the case that we don't know exactly the number of each groups by A P we have

$$\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_1 \geq 1 \\ \lambda_2, \lambda_3 \geq 0}} \lambda_1 \binom{n}{\lambda_1 \ \lambda_2 \ \lambda_3}$$

RHS: first we choose one ball, paint it to red and then stack ($*$) on it, [n ways to do so] and then we have $n-1$ balls, for each 3 possible colore $\Rightarrow$ by M.P. $n \cdot 3^{n-1}$ ✓

c) $$\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_1, \lambda_2, \lambda_3 \geq 0}} \lambda_1 \lambda_2 \binom{n}{\lambda_1 \lambda_2 \lambda_3} = n(n-1)\, 3^{(n-1)}$$

$$\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_i \geq 0}} \lambda_1\lambda_2 \binom{n}{\lambda_1\ \lambda_2\ \lambda_3} = \sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_1,\lambda_2\geq 1 \\ \lambda_3 \geq 0}} \lambda_1\lambda_2 \binom{n}{\lambda_2\ \lambda_1\ \lambda_3}$$

LHS: the same as before first we divide n balls into 3 coloring groups and paint them then we choose one red ball and stack ($*$) on it and one blue ball and stack ( ) on it $\Rightarrow \binom{n}{\lambda_1\ \lambda_2\ \lambda_3}\lambda_1\lambda_2$ then we have to consider all possible partitions of balls into 3 coloring groups by A.P.: $\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_1,\lambda_2\geq 1 \\ \lambda_3 \geq 0}} \lambda_1\lambda_2 \binom{n}{\lambda_1\ \lambda_2\ \lambda_3}$

RHS: first we choose one ball and paint it into red and stack ($*$) on it followed by choosing one ball from $n-1$ remaining ball and paint it into blue and stack ( ) on it! it remains to paint $n-2$ balls, each ball have 3 possible choice of colloring $\Rightarrow$

$$\underset{\substack{\text{red}\\(*)}}{n}\ \underset{\substack{\text{blue}\\(\ )}}{(n-1)}\ 3^{n-2} \quad \underbrace{\begin{matrix}\text{red}\\\text{blue}\\\text{green}\end{matrix}\ \cdots\ \begin{matrix}\text{red}\\\text{blue}\\\text{green}\end{matrix}}_{n-2}$$

✓

To be resubmitted

$\boxed{\text{Ex. 3}}$ D_n is the # of derangements of $[n]$. Show that D_n satisfies:

$$D_n = n D_{n-1} + (-1)^n.$$

- $D_n \underset{\substack{\uparrow \\ \text{proved} \\ \text{in lecture}}}{=} (n-1) D_{n-1} + (n-1) D_{n-2}$.

$$\overbrace{D_n - n D_{n-1}}^{F(n)} = (-1) D_{n-1} + (n-1) D_{n-2} = (-1)\overbrace{\left(D_{n-1} - (n-1) D_{n-2}\right)}^{F(n-1)}$$

observe we obtain a expression of the form $F(n) = (-1) F(n-1)$

thus $F(n) = (-1) F(n-1) = \dots = (-1)^{n-1} F(2) = (-1)^{n-1} (\underset{0}{D_1} - \underset{1}{D_0}) =$

$= (-1)^n$.

- $F(n) = D_n - n D_{n-1} \overset{=(-1)^n}{} \Rightarrow D_n = (-1)^n + n D_{n-1}$

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Problem 5

(Martin)

a) $R_n = R_{n-1} + 6R_{n-2}$, where $R_0 = 0, R_1 = 1$.

$$\rightsquigarrow \chi(R_n) = x^2 - x - 6 = (x-3)(x+2)$$

So the general solution is $R_n = a3^n + b(-2)^n$. Calculate the coefficients:

$$\begin{array}{rcrcrcr} R_0 & = & 0 & = & a & + & b \\ R_1 & = & 1 & = & 3a & - & 2b \end{array}$$

Solving the linear system $\rightsquigarrow a = \frac{1}{5},\ b = -\frac{1}{5}$.

$$R_n = \frac{1}{5}3^n - \frac{1}{5}(-2)^n = \frac{1}{5}\Big(3^n - (-2)^n\Big)$$

is a closed from for the given recurrence relation.

b) $R_n = 5R_{n-1} + 29R_{n-2} - 105R_{n-3}$, where $R_0 = 0, R_1 = 1, R_2 = 1$.

$$\rightsquigarrow \chi(R_n) = x^3 - 5x^2 - 29x + 105$$

Guessing one root, find the root 3:

$$\chi(R_n) = (x-3)(x^2 - 2x - 35) = (x-3)(x+5)(x-7)$$

So the general solution is $R_n = a3^n + b(-5)^n + c7^n$. Calculate the coefficients:

$$\begin{array}{rcrcrcrcr} R_0 &=& 0 &=& a &+& b &+& c \\ R_1 &=& 1 &=& 3a &-& 5b &+& 7c \\ R_2 &=& 1 &=& 9a &+& 25b &+& 49c\,. \end{array}$$

Solving the linear system leads to $a = \frac{1}{32}$, $b = -\frac{3}{32}$, $c = \frac{1}{16}$.

$$R_n = \frac{1}{32}3^n - \frac{3}{32}(-5)^n + \frac{1}{16}7^n = \frac{1}{32}\left(3^n - 3\cdot(-5)^n + 2\cdot 7^n\right)$$

is a closed form for the given recurrence relation.

c) $R_n = 8R_{n-1} - 21R_{n-2} + 18R_{n-3}$, where $R_0 = 0, R_1 = 0, R_2 = 1$.

$$\rightsquigarrow \chi(R_n) = x^3 - 8x^2 + 21x - 18$$

Guessing one root, find the root 2:

$$\chi(R_n) = (x-2)(x^2 - 6x + 9) = (x-2)(x-3)^2$$

So the general solution is $R_n = a2^n + b3^n + cn3^n$. Calculate the coefficients:

$$\begin{array}{rcrcrcrcr} R_0 &=& 0 &=& a &+& b && \\ R_1 &=& 0 &=& 2a &+& 3b &+& 3c \\ R_2 &=& 1 &=& 4a &+& 9b &+& 18c \end{array}$$

Solving the linear system leads to $a = 1, b = -1, c = \frac{1}{3}$.

$$R_n = 2^n - 3^n + \frac{1}{3}n3^n = 2^n + 3^{n-1}(n-3)$$

is a closed form for the given recurrence relation.

Mattes Mollenhauer

Problem 8

Claim

$$\sum_{k=0}^{n} \binom{n}{k} D_k = n!$$

Proof Combinatorial.

LHS. By MP,

$\binom{n}{k} D_k$ counts the

ways to choose k elements out of $[n]$ and then permuting these k elements without fixpoints.

and $\binom{n}{k} D_k$ is the number of

permutations with $n-k$ fixpoints over $[n]$.

By AP, summing over all $k \in \{0, \dots, n\}$

we get all possible permutations of $[n]$.

RHS. Counts the number of permutations on $[n]$. (see lecture)

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$$\Rightarrow \sum_{k=0}^{n} \binom{n}{k} D_k = n!$$

#10

a)

Our relations starts with three thousand for $n = 0$ since this represents year 1995. We take the value of the last year 1.02 times since we have a increase of 2% of the salary depending on the last year.

$F_0 = 3000$
$F_n = 1.02F_{n-1} + 100$

b)

We solve the recurrence relation by iteration:
$F_n = 1.02F_{n-1} + 100$
$= 1.02(1.02F_{n-2} + 100) + 100$
$= 1.02(1.02(1.02F_{n-3} + 100) + 100) + 100$
$= 1.02^3 F_{n-3} + 1.02^2 * 100 + 1.02^1 * 100 + 1.02^0 * 100$

After iterating n times this leads to:

$1.02^{n} F_{n-n} + 1.02^{n-1} * 100 + 1.02^{n-2} * 100 + \cdots + 100$
$= 1.02^n F_{n-n} + \sum_{i=0}^{n-1} 1.02^i * 100$
$= 1.02^n F_{n-n} + 100 * \sum_{i=0}^{n-1} 1.02^i$
$= 1.02^n F_{n-n} + 100 * \frac{1.02^n - 1}{1.02 - 1} = 1.02^n * 3000 + 100 * \frac{1.02^n - 1}{1.02 - 1}$

Therefore the closed form is $1.02^n * 3000 + 100 * \frac{1.02^n - 1}{1.02 - 1}$.
I made the sanity check, whether the result of both forms are equal for $n \in \{1, \ldots, 1000\}$ (not by hand).

c)

We have to calculate our relation for $n = 2017 - 1995 = 22$. The result of calculation of the closed form as well as the recurrence relation is:
7367.83736620703885963048959406693170787909632