Exercise Sheet 4

Discrete Mathematics I - SoSe17

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Due date 17 May 2017 -- 16:00

You should try to solve all of the exercises below, and select three to four solutions to be submitted and graded. We encourage you to submit in pairs, please remember to indicate the author of each individual solution and the name of both team members on the cover sheet.

Problem 1

Find the number of lattice paths from $(0,0,0,0)$ to $(7,5,3,9)$ that pass through $(5,3,1,5)$ but not $(4,2,0,3)$.

Problem 2

Prove the following identities:

a) $\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n\\ \lambda_1,\lambda_2,\lambda_3\geq 0}} \binom{n}{\lambda_1,\lambda_2,\lambda_3} = 3^n$,

b) $\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n\\ \lambda_1,\lambda_2,\lambda_3\geq 0}} \binom{n}{\lambda_1,\lambda_2,\lambda_3}\lambda_1 = n3^{n-1}$,

c) $\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n\\ \lambda_1,\lambda_2,\lambda_3\geq 0}} \binom{n}{\lambda_1,\lambda_2,\lambda_3}\lambda_1\lambda_2 = n(n-1)3^{n-2}$.

Problem 3

Let D_n be the number of dérangements of $[n]$. Show that the sequence $(D_n)_{n\in\mathbb{N}}$ satisfies the recurrence relation

$$D_n = n \cdot D_{n-1} + (-1)^n,$$

with initial condition $D_0 = 1$.

Problem 4

A triangulation of a regular n-gon is a way of placing $n-3$ non-intersecting diagonals in the n-gon. Give a recurrence with appropriate initial conditions for T_n, the number of triangulations on the regular n-gon. Identify the recurrence as completely as possible.

Problem 5

Find a closed form for the recurrence relations on $(R_n)_{n\in\mathbb{N}}$

a) $R_n = R_{n-1} + 6R_{n-2}$, where $R_0 = 0$ and $R_1 = 1$,
b) $R_n = 5R_{n-1} + 29R_{n-2} - 105R_{n-3}$, where $R_0 = 0$, $R_1 = 1$ and $R_2 = 1$,
c) $R_n = 8R_{n-1} - 21R_{n-2} + 18R_{n-3}$, where $R_0 = R_1 = 0$, and $R_2 = 1$.

Problem 6

Find the general solution for the recurrence

$$R_{n+5} - 9R_{n+4} + 31R_{n+3} - 63R_{n+2} + 108R_{n+1} - 108R_n = 0.$$

Problem 7

Show that among any 5 points on a plane, no three points through a line, 4 of them must form a convex quadrangle.

Problem 8

Show, using a combinatorial argument that

$$\sum_{k=0}^{n} \binom{n}{k} D_k = n!,$$

where D_k is the number of dérangements of $[k]$.

Problem 9

Say if the given sequences are particular solutions of the recurrence relation $A_n = 8A_{n-1} - 16A_{n-2}$.

a) $(0)_{n\in\mathbb{N}}$,
b) $(2^n)_{n\in\mathbb{N}}$,
c) $(4^n)_{n\in\mathbb{N}}$,
d) $(n2^{2n})_{n\in\mathbb{N}}$,
e) $(2 \cdot 4^n + 3n4^n)_{n\in\mathbb{N}}$,
f) $(n^2 4^n)_{n\in\mathbb{N}}$,

Problem 10

A worker was hired in 1995 with 3 000Euros/month. Every year the worker receives an increase of 100Euros plus 2% of the monthly salary of the previous year.

a) Give a recurrence relation to compute the monthly salary at year $1995 + n$.
b) Solve this recurrence relation.
c) Compute the monthly salary in 2017.