Exercise Sheet 3 (Partial Solutions)

Discrete Mathematics I - SoSe17

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Due date 10 May 2017 -- 16:00

You should try to solve all of the exercises below, and select three to four solutions to be submitted and graded. We encourage you to submit in pairs, please remember to indicate the author of each individual solution.

Problem 1

Show that the number of permutations of $[n]$ is

$$\sum_{(b_1,\ldots,b_n)} \frac{n!}{b_1!\cdots b_n! 1^{b_1}\cdots n^{b_n}}$$

where $\sum_{i=1}^{n} ib_i = n$.

Problem 2

Give a bijection between the subsets of $[n-1]$ and the compositions of n.

Consider the string $\bullet|_1 \bullet |_2 \bullet |_3 \cdots |_{n-1}\bullet$. Given a subset S of $[n]$, remove the separators $|_i$ such that $i \in S$ to obtain a shorter string. Then read the string from left to right and write the number of bullets $\bullet$ between the separators that remain. This gives a composition of n. This map is surjective from a direct construction: remove the separators needed to obtain the desired composition. Further, there is a unique way to get a specific composition, i.e. this is also injective. Thus is map is bijective.

Problem 3

What is the number of compositions of n into odd parts?

Problem 4

Give a combinatorial proof that the number of set-partitions of $[n]$ such that no two consecutive integers appear in the same block is the Bell number $B(n-1)$.

Let π be a set partition of $[n-1]$. Consider a maximal sequence $i, i+1, \ldots, j$ of consecutive integers in a block of π with $j > i$. Remove $j-1, j-3, \ldots$ from this block and put them into a new block along with n. Repeat this procedure until there are no consecutive integers in a block.

You obtain a set-partition of :math:‘[n]‘ without two consecutive integers in the same block. To reverse this map, take any set partition of $[n]$. Look for the maximal element in the block of n, say a, and all its elements that form an arithmetic sequence with difference 2 in the block of n. By assumption, $a-1$ is not in that block. In the block of $a-1$, look for the longest decreasing arithmetic sequence with difference 2; say it ends with $a-k$. Then, take a, $a-2$, etc. and put them in the block of $a-1$ until you adjoint $a-k-1$ (if you can). Continue until the block of n only contains n. You will obtain the set partition π of $[n-1]$ you started with. Hence this map is reversible, and thus gives a bijection.

Problem 5

Let P be a convex polygon with n sides. We add all $\binom{n}{2}$ diagonals and we assume that no three intersect in one point. If we remove all intersection point (and the vertices of P) how many line segments are left?

Problem 6

In how many way can we separate 23 people in three groups of 2, 1 group of 3, 1 group of 4, and two groups of 5?

The groups are not distinguishable (they are not mentionned to be named or labeled). Therefore, we first count the number of ways to forms sequencially 3 subsets of 2, 1 subset of 3, one subset of 4, and two subsets of 5 from the set [23]. This is given by the multinomial coefficient $\binom{23}{2,2,2,3,4,5,5} = \frac{23!}{2!2!2!3!4!5!5!} = \frac{23!}{2^{13}3^{4}5^{2}}$. The multinomial coefficient takes a specific ordering into account when creating the groups. Since groups are not distinguishable, we divide the obtained number by "permuting" the groups with the same cardinality: 3! (for the groups of 2) and 2! (for the groups of 5). Thus we obtain: $\frac{2^{19}\cdot 3^{9}\cdot 5^{4}\cdot 7^{3}\cdot 11^{2}\cdot 13\cdot 17\cdot 19\cdot 23}{2^{15}3^{5}5^{2}} = 2^4\cdot 3^4\cdot 5^2\cdot 7^3\cdot 11^2\cdot 13\cdot 17\cdot 19\cdot 23$ ways to form the groups.

Problem 7

In how many ways can we distribute

a) 5 unlabeled objects in 3 unlabeled boxes?

b) 5 unlabeled objects in 3 labeled boxes?

c) 5 labeled objects in 3 unlabeled boxes?

d) 5 labeled objects in 3 labeled boxes?

Problem 8

What are the number of integer solutions to the following (in)equalities?

a) $x_1 + x_2 + x_3 + x_4 = 17$ and $x_i \geq 0$,

b) $x_1 + x_2 + x_3 + x_4 + x_5 = 21$ and $x_i \geq 1$,

Since all variables have to be at least 1 we can remove 5 from 21 and get an equivalent system $y_1 + y_2 + y_3 + y_4 + y_5 = 16$, where $y_i \geq 0$.

Now, we have to distribute 16 undistinguishable units into 5 distinguishable (ordered) variables. This can be done by ordering 16 one's and 4 separators "|": we need to choose where, among the 20 symbols, the separators will appear. This is equal to $\binom{20}{4} = 3\cdot 5\cdot 17\cdot 19$. Hence there are 4845 solutions.

c) $x_1 + x_2 + x_3 + x_4 = 20$ and $0 < x_1 < 5, x_2 > 5, 0 < x_3 < 6, x_4 > 0$,
First, we allocate the mandatory units into the variables to get an equivalent system $y_1+y_2+y_3+y_4 = 11$ where $y_1 = x_1-1, y_2 = x_2-6, y_3 = x_3-1, y_4 = x_4-1$. The inequalities become $0 \leq y_1 \leq 3, y_2 \geq 0, 0 \leq y_3 \leq 4$ and $y_4 \geq 0$. As above, the number of nonnegative solutions is therefore $\binom{14}{3} = 364$. But we have to remove the solution where $y_1 \geq 4$ (Set A) or $y_3 \geq 5$ (Set B). Using the Inclusion-Exclusion principle in order to compute the union of the solutions in Set A and Set B, we get $|A|+|B|-|A \cap B| = \binom{10}{3}+\binom{9}{3}-\binom{5}{3} = 194$. Hence the number of solutions is 364-194=170.

d) $x_1 + x_2 + x_3 + x_4 \leq 20$ and $x_i \geq 0$,

e) $x_1 + x_2 + x_3 + x_4 \leq 20$ and $-2 \leq x_1 \leq 8, -2 \leq x_2 \leq 8,\ x_3, x_4 \geq 1$,

f) $ax_1 + x_2 + x_3 = an$ and $a, n \in \mathbb{N}, x_i \geq 0$.

Problem 9

An investor has 20 000 Euros to invest in four different funds. Knowing that he will invest an integer number of kEuros (1000Euros) in each fund, what is the number of strategies if

a) the total 20 000 Euros is invested?

b) the total 20 000 Euros is invested in at least 2 funds?

c) only a part of the money could be invested?

Problem 10

Give a bijection between the set of integer partitions of n into at most k parts and the set of integer partitions of n whose parts do not exceed k.