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Discrete Maths I, Exercise Sheet 3 — Florian Zeidler — $\frac{3}{3}$

**Ex. 7** MH

3 ★ nice!

a) The 12-fold way gives us for 5 balls (u) and 3 boxes (u):

$$\sum_{i=1}^{3} P_{5,i} = P_{5,3} + P_{5,2} + P_{5,1} = (\underbrace{P_{2,3}}_{=0} + P_{4,2}) + (P_{3,2} + P_{4,1}) + (P_{4,1} + \underbrace{P_{4,0}}_{=0})$$

$$= P_{4,2} + P_{3,2} + 2 \cdot P_{4,1} = (P_{2,2} + \underbrace{P_{3,1}}_{=P_{2,1}}) + (\underbrace{P_{1,2}}_{=0} + P_{2,1}) + 2 \cdot (\underbrace{P_{3,1}}_{=P_{2,1}} + \underbrace{P_{3,0}}_{=0})$$

$$= P_{2,2} + 4 \cdot P_{2,1} = P_{1,1} + 4(P_{1,1} + \underbrace{P_{1,0}}_{=0}) = 5(\underbrace{P_{0,1}}_{=0} + \underbrace{P_{0,0}}_{=1}) = 5 \text{ ways}$$

b) For 5 balls (u) and 3 boxes (l) the 12-fold way gives us:

$$\binom{5+3-1}{3-1} = \binom{7}{2} = \frac{7!}{5!\,2!} = \frac{7 \cdot 6}{2} = 7 \cdot 3 = 21 \text{ ways.}$$

c) For 5 balls (l) and 3 boxes (u) the 12-fold way gives us:

$$\sum_{i=1}^{3} S_{5,i} = S_{5,3} + S_{5,2} + S_{5,1} = (3 \cdot S_{4,3} + S_{4,2}) + (2 \cdot S_{4,2} + S_{4,1}) + (S_{4,1} + \underbrace{S_{4,0}}_{=0})$$

$$= 3S_{4,3} + 3S_{4,2} + 2S_{4,1} = 3(S_{3,3} \cdot 3 + S_{3,2}) + 3(S_{3,2} \cdot 2 + S_{3,1}) + 2(S_{3,1} + \underbrace{S_{3,0}}_{=0})$$

$$= 9\underbrace{S_{3,3}}_{=1} + 9S_{3,2} + 5S_{3,1} = 9 + 9(\underbrace{S_{2,2}}_{=1} \cdot 2 + S_{2,1}) + 5(S_{2,1} + \underbrace{S_{2,0}}_{=0})$$

$$= 9 + 18 + 14S_{2,1} = 27 + 14(\underbrace{S_{1,1}}_{=1} + \underbrace{S_{1,0}}_{=0}) = 41 \text{ ways}$$

d) For 5 Balls (l) and 3 boxes (l) the 12-fold way yields:

$$3^5 = 243 \text{ ways}$$

Remark: Obviously the functions here are arbitrary; In the exercise there are no restrictions for the functions! ☺

MH 3 beautiful!

$\boxed{Ex. 8}$ a) $x_1 + x_2 + x_3 + x_4 = 17, \ x_i \geq 0$

This is the same problem as distributing 17 red balls in 4 labeled boxes. $\rightarrow \binom{17+4-1}{4-1} = \binom{20}{3} = 1140$ ways/int. solutions

b) $x_1 + x_2 + x_3 + x_4 + x_5 = 21, \ x_i \geq 1$

has the same amount of integer solutions as

$$x_1 + x_2 + x_3 + x_4 + x_5 = 16, \ x_i \geq 0$$

$$\rightarrow \binom{16+5-1}{5-1} = \binom{20}{4} = 4845 \text{ integer solutions}$$

c) $x_1 + x_2 + x_3 + x_4 = 20, \ 0 < x_1 \leq 5, \ x_2 > 5, \ 0 < x_3 < 6, \ x_4 > 0$

has the same amount of int. solutions as

$$x_1 + x_2 + x_3 + x_4 = 11, \ 3 \geq x_1 \geq 0, \ x_2 \geq 0, \ 4 \geq x_3 \geq 0, \ x_4 \geq 0$$

Let us now consider
I $x_1 + x_2 + x_3 + x_4 = 11, \ x_i \geq 0$
II $x_1 + \dots + x_4 = 11, \ x_1 \geq 4, \ x_{2,3,4} \geq 0$
III $x_1 + \dots + x_4 = 11, \ x_3 \geq 5, \ x_{1,2,4} \geq 0$
IV $x_1 + \dots + x_4 = 11, \ x_3 \geq 5, \ x_1 \geq 4, \ x_{2,4} \geq 0$

with
II $\leftrightarrow$ $x_1 + \dots + x_4 = 7, \ x_i \geq 0$
III $\leftrightarrow$ $x_1 + \dots + x_4 = 6, \ x_i \geq 0$
IV $\leftrightarrow$ $x_1 + \dots + x_4 = 2, \ x_i \geq 0$

It is now obvious, that #IntSol. = #I − (#II + #III) + #IV (with the Inclusion/Exclusion Principle).

$$\Rightarrow \#\text{IntSol.} = \binom{11+4-1}{4-1} - \binom{7+4-1}{4-1} - \binom{6+4-1}{4-1} + \binom{2+4-1}{4-1}$$

$$= \binom{14}{3} - \binom{10}{3} - \binom{9}{3} + \binom{5}{3}$$

$$= 170$$

Matthias Himmelhuber
Florian ...

d) $x_1+x_2+x_3+x_4 \leq 20$ and $x_i \geq 0$

For all possible values on the RHS from 0 to 20, we need to check the possible amount of integer solutions

$\xrightarrow{AP}$ $\sum_{k=0}^{20} \binom{k+4-1}{4-1} = \sum_{k=0}^{20} \binom{k+3}{3} = 10{,}626$ integer solutions

e) $x_1+x_2+x_3+x_4 \leq 20$ and $-2 \leq x_1 \leq 8$, $-2 \leq x_2 \leq 8$, $x_3, x_4 \geq 1$

has the same amount of integer solutions as

$x_1+x_2+x_3+x_4 \leq 22$, $0 \leq x_1 \leq 10$, $0 \leq x_2 \leq 10$, $x_3, x_4 \geq 0$

Consider
I $x_1+\dots+x_4 \leq 22$, $x_1 \geq 11$, $x_{2,3,4} \geq 0$
II $x_1+\dots+x_4 \leq 22$, $x_2 \geq 11$, $x_{1,3,4} \geq 0$
III $x_1+\dots+x_4 \leq 22$, $x_{1,2} \geq 11$, $x_{3,4} \geq 0$
IV $x_1+\dots+x_4 \leq 22$, $x_{1,2,3,4} \geq 0$

with II, I $\leftrightarrow$ $x_1+\dots+x_4 \leq 11$, $x_i \geq 0$
III $\leftrightarrow$ $x_1+\dots+x_4 \leq 0$, $x_i \geq 0$

With the help of (c)+(d) this gives us (+incl./excl. principle)

$$\#\text{IntSol.} = \#\text{IV} - 2\#\text{I} + \#\text{III}$$
$$= \sum_{k=0}^{22} \binom{k+3}{3} - 2 \cdot \sum_{k=0}^{11} \binom{k+3}{3} + 1$$
$$= 14950 - 2 \cdot 1365 + 1$$
$$= 12{,}221$$

f) $ax_1+x_2+x_3 = an$, $x_i \geq 0$ and $a, n \in \mathbb{N}$.

Firstly, we observe what happens, when picking $x_1$ first:
$x_1$ can be picked from 0 to $n^*$. For $x_1 = 0$, we get $\binom{an+2-1}{2-1} = \binom{an+1}{1} = an+1$ choices for $x_2+x_3 = an$,

For $x_1 = 1$, it should hold that $x_2+x_3 = a(n-1)$, $x_i \geq 0$.
This gives us $\binom{a(n-1)+1}{1} = a(n-1)+1$ choices. We do this until $x_1 = n$, so that $x_2+x_3 = 0$, $x_i \geq 0 \rightarrow \binom{1}{1} = 1$ choices. For

$*$ so that the equality makes sense.

Since the integer solutions for different picks of $x_1$ are disjoint, we just have to sum up all the integer solutions for $x_{2,3}$, depending on $x_1$.

$$\Rightarrow \# \text{Int Sol.} = (an+1) + (a(n-1)+1) + \dots + (a+1) + (0+1)$$
$$= \sum_{k=0}^{n} (ak+1) = a \cdot \sum_{k=0}^{n} k + (n+1)$$
$$= \frac{a}{2}(n+1)n + (n+1) = \left(\frac{an+2}{2}\right)(n+1)$$

Small Gauss Theorem

$$= \frac{1}{2}(n+1)(an+2)$$

2

3 $\boxed{\text{Ex. 9}}$ a) Since we only want to invest integer amounts of money,

FZ this is the same as viewing $x_1 + x_2 + x_3 + x_4 = 20$, $x_i \geq 0$

(translation of MH) which has $\binom{20+4-1}{4-1} = \binom{23}{3} = 1{,}771$ ways of investment.

that explains the same handwriting. You are also welcome to submit in German. (and possibly French, but I think that is the end of the list)

b) $x_1 + x_2 + x_3 + x_4 = 20$, $x_{1,2} \geq 1, x_{3,4} \geq 0 \vee x_{1,3} \geq 1, x_{2,4} \geq 0$, etc.

We can now view

I $x_1 + \dots + x_4 = 20$, $x_i \geq 0$ and II $x_1 + \dots + x_4 = 20$, $x_1 = 20 \vee x_2 = 20 \vee x_3 = 20 \vee x_4 = 20$, rest 0

where it is only invested in one fund

Now we see that $\# \text{Inv. Poss.} \overset{w/Ex}{=} \# \text{I} - \# \text{II}$

$$= \binom{23}{3} - 4 = 1{,}767 \text{ ways of investing}$$

c) $x_1 + x_2 + x_3 + x_4 \leq 20$, $x_i \geq 0$

With ex. 8 this gives us

$$\sum_{k=0}^{20} \binom{k+4-1}{4-1} = \sum_{k=0}^{20} \binom{k+3}{3} = 10{,}626 \text{ strategies}$$

D 3

## Problem 6

[Kelvin] good!

The main problem in this exercise is, that we have indistinguishable (e.g. the three groups with two people) and distinguishable groups (e.g. we can distinguish between a group with four or five people). Therefore we are using two steps to calculate all combinations:

The first step is to sum up duplicate groups (as the group of two people) and determine the number of combinations to choose these groups: $\binom{23}{3} * \binom{20}{6} * \binom{14}{4}$
*
The naive approach is to count the combinations of these subgroups like $\binom{6}{2} * \binom{4}{2}$ for the groups of two and $\binom{10}{5}$ for the groups of five people. But this would be wrong if you just multiply these values with our first result, since we should not distinguish the groups of two people (or five people), because these groups are unlabeled. 3! (and 2!) ist the number of permutations if

*In the second step we have to make sure, that we count the combinations of the subgroups (~~and~~ e.g. three subgroups of two people in the group of six) unlabeled.

we distinguish the groups. We count the combinations of the groups of two people 3! times to often (and the groups of five 2! to often). This leads to $\binom{23}{3} * \binom{20}{6} * \binom{14}{4} * \binom{6}{2} * \binom{4}{2} * (3!)^{-1} * \binom{10}{5} * (2!)^{-1} =$
$129866821.5 * 10^6$

~~7~~ 3

## Problem 9

**[Kelvin]**

### a)

We can formulate the problem as $\sum_{i=1}^{4} f_i = 20$
We have to select "No,Yes,No,No" in the "Twelvefold Way"-Table, since the money (balls) is unlabeled (I:No) and the funds are labeled (II:Yes). Fonds without an investment are allowed (a: No) and an investment can be higher than $1k$ (b: No). As in the example, this leads us to $\binom{20+4-1}{4-1} = 1771$

### b)

We are searching for combinations with at least two balls ($1k$ of investment) in different boxes (funds). We use the A.P. to sum up the cardinality of the sets of combinations with "Investment in exactly two funds", "Investment in exactly three funds" and "Investment in exactly four funds". Since there are also different ways to choose the funds and these are distinguishable, we have to multiply the cardinality with the number of possible combinations to choose. The cardinality of the set of all combinations of "Investment in exactly $a$ funds"

is equal to the number of surjective functions (a: Yes, b: No) from unlabeled (I:No) balls ($20k$ possible investment) to labeled (II:Yes) boxes ($a$ funds). This leads to $\sum_{i=2}^{4} \binom{4}{i} \binom{20-1}{i-1} = 6 * \binom{19}{1} + 4 * \binom{19}{2} + 1 * \binom{19}{3} = 1767.$ ✓
As i recognized at last it would be much simpler to count the number of invalid combinations (with and investment for exactly one fund) and substract it from the solution of subproblem 9.a. Since we are only able to invest in exactly one fund in four ways, the solution is obvious. ok well, same number ☺

## c)

We are interested in all combinations, where we invest "a part of the money". We assume that a part means a number $a \in \mathbb{N}$ with $1 \leq a \leq 19$.
We have to sum up all combinations from the investment of $1k$ to $19k$. We use the same approach as in 9.a and this leads to $\sum_{i=1}^{19} \binom{i+4-1}{4-1} = 8854$ ✓

Problem 7 (Sophia Elia) Perfect! 3

In how many ways can we distribute 5 (un)labeled objects in 3 (un)labeled boxes?

This problem is a direct application of the "12 ways" we learned in class, and I will simply cite results there to calculate ways in this problem. As its unspecified, I assume the way we distribute objects is arbitrary. ok.

ⓐ 5 unlabeled objects in 3 unlabeled boxes:

There are $\sum_{i=1}^{3} P_{5,i}$ ways to arbitrarily place 5 unlabeled objects in 3 boxes.

Here $P_{5,i}$ is the number of partitions of 5 into $i$ parts. We saw in class that $P_{n,k}$ satisfies the following recurrence relation:

$P_{0,0}=1 \quad P_{n,k}=0$ if $n<0$ or $k<0$ $\quad P_{n,k}=P_{n-k,k}+P_{n-1,k-1}$

I will make a table in order to calculate $P_{5,i}$

value of $i$

| value of $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 1 | 0 | 0 | 0 |
| 3 | 0 | 1 | 1 | 1 | 0 | 0 |
| 4 | 0 | 1 | 2 | 1 | 1 | 0 |
| 5 | 0 | 1 | 2 | 2 | 1 | 1 |

Thus there are

$P_{5,1}+P_{5,2}+P_{5,3}$

$=1+2+2$

$=5$ ways to arbitrarily place 5 unlabeled objects in 3 unlabeled boxes

note: it's impossible to partition a set with a positive number of elements into 0 parts

Problem 7 continued

(b) Place 5 unlabeled objects in 3 labeled boxes in an arbitrary way.

The number of ways to do this, by the twelvefold ways, is

$$\binom{5+3-1}{3-1} = \binom{7}{2} = \frac{7!}{(7-2)!\,2!} = \frac{7 \cdot 6}{2} = 21 \text{ ways.}$$

(c) Place 5 labeled objects in 3 unlabeled boxes

By the twelvefold ways, there are $\sum_{i=1}^{k} S_{5,i}$ ways to do this.

$S_{n,i}$ is the "Sterling Number of the Second Kind", and represents set partitions of $[n]$ into $i$ blocks. There is a reccurence relation for computing:

$S_{0,k} = 1$ $S_{n,k} = 0$ if $n<k$ and $S_{n,k} = k \cdot S_{n-1,k} + S_{n-1,k-1}$.

Again I make a tabe to compute

| value of n \ value of k | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 |
| 2 | 0 | 1 | 1 | 0 |
| 3 | 0 | 1 | $2+1$ 3 | 1 |
| 4 | 0 | $1+0$ 1 | $2 \cdot 3+1$ 7 | $3+3$ 6 |
| 5 | 0 | 1 | $2 \cdot 7+1$ 15 | $3 \cdot 6+7$ 25 |

There are thus

$$\sum_{i=1}^{k} S_{5,i} = 1+15+25 = 41 \text{ ways}$$

to put 5 labeled objects in 3 unlabeled boxes.

7 continued

(d) Place 5 labeled objects in 3 labeled boxes in an arbitrary way.

There are $3^5 = 243$ ways to do this by the twelvefold ways. ✓

3'

Problem $\boxed{6}$: In how many ways can we separate 23 people into 3 groups of 2, 1 group of 3, 1 group of 4, and 2 groups of 5.

We will count the ways using steps and the multiplication principle.

Step 1: Choose the group of 3. This is done in $\binom{23}{3}$ ways.

2. Follow this by choosing the group of 4 from the remaining 20 people. This is done in $\binom{20}{4}$ ways.

3. Follow this by choosing 6 people that will get split into groups of 2 from the remaining 16 people. Do this in $\binom{16}{6}$ ways. ✓

We now count the ways to split them into groups of 2:
- choose first team in $\binom{6}{2}$ ways. Then choose next team in $\binom{4}{2}$ ways. Then divide by the ways to reorder the 3 teams $= 3!$ because we don't care in which order they're chosen. ✓

4. Finally, of the ten people remaining, choose five of them to be in a group. This forces the remaining people to be the second group of 5. This is done in $\binom{10}{5}$ ways. Divide by 2! b/c we don't care which group is which. ✓

By the multiplication principle, there are $\binom{23}{3}\binom{20}{4}\binom{16}{6}\frac{\binom{6}{2}\binom{4}{2}}{3!}\frac{\binom{10}{5}}{2!} \approx 1.2\times10^{14}$ ✓