Exercise Sheet 2 (Partial Solutions)

Discrete Mathematics I - SoSe17

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Due date 3 May 2017 -- 16:00

You should try to solve all of the exercises below, and select three to four solutions to be submitted and graded. We encourage you to submit in pairs, please remember to indicate the author of each individual solution.

Problem 1

Prove combinatorially the following equality

$$n\binom{n-1}{k-1} = \binom{n}{k}k.$$

Task: From a group of n people, form a committee of k people and assign a committee president.

1) The LHS counts the ways to do the task by first choosing the committee president out of n people followed by choosing $k-1$ committee members out of the $n-1$ people left.
2) The RHS counts the ways to do the task by first choosing the k committee members followed by choosing the president out of the k members.

Problem 2

Prove combinatorially the following equality

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Problem 3

Place n points on a circle and draw all line segments joining pairs of of these points and assume that no three line segments intersect in one point inside the circle. The line segments determine bounded regions inside the circle.

a. How many regions are determined by $n = 1, 2, 3, 4, 5, 6$ points?

b. Prove combinatorially (using a bijective proof), that the number of regions is

$$\binom{n-1}{0}+\binom{n-1}{1}+\binom{n-1}{2}+\binom{n-1}{3}+\binom{n-1}{4}.$$

Hint: Label the vertices clockwise and label the regions using 0, 1, 2, 3, or 4 vertex labels from $n-1$ labels.

a) 1,2,4,8,16,31

b) Label the n vertices clockwise from 0 to $n-1$. There are two types of regions inside the circle: i) those whose boundary do not intersect the circle and ii) those who do.

Consider $0 \le a < b < c < d \le n-1$. The chords ac and bd intersect inside the circle. Going from the intersection point towards d, we label the region located to the left by $abcd$. If $a = 0$, then we drop the label a to get bcd. All regions of type i) are labeled this way; their vertices respect the criterion on a, b, c, d and it receives only one label. Hence we have labeled type i) regions with $\binom{n-1}{3}+\binom{n-1}{4}$ labels.

Consider $0 \le a < b \le n-1$. There is a unique region located to the left of the segment ab when going from vertex a to vertex b. Label this region by ab. If $a = 0$, then drop the label a to get just b. All regions that have a vertex on the boundary are labeled with exactly one label. Finally, the region determined by the arc between 0 and $n-1$ and the segment between them receives no label.

Every regions thus receives a unique label consisting of 0, 1, 2, 3, or 4 distinct numbers from 1 to $n-1$.

Problem 4

Prove that the number of regions inside the circle in Problem 3 is

$$1+\binom{n}{2}+\binom{n}{4}$$

using an inductive argument (and not the arguments or proof from the previous problem).

Problem 5

Let $n \geq 2$ and $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ its prime factorization. Show that its number of divisors is

$$\prod_{i=1}^{k}(a_i+1).$$

Problem 6

Prove that there exists a power of 3 that ends with "001".

Take 1001 distinct powers of 3. By the Pigeonhole Principle, two of them must be congruent modulo 1000; say $3^a \equiv 3^b \mod 1000$, with $b > a$. Therefore, we have $3^a(3^{b-a}-1) \equiv 0 \mod 1000$. Since 3 and 1000 are coprime, 1000 must divide $3^{b-a}-1$. Thus 3^{b-a} ends with "001".

Problem 7

Show that at a party with at least 2 people, there are 2 people that know exactly the same number of people at the party. (We assume that knowing is reflexive and symmetric: A knows A and if A knows B, then B knows A)

Problem 8

Prove the Vandermonde identity using an algebraic argument

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}.$$

Problem 9

What is the coefficient of

a. x^9 in $(2-x)^{19}$?

b. x^k in $(x+1/x)^{100}$?

c. $x^{50}y^{40}$ in $(x+y-2/x)^{100}$?

a)

$$\left[(2-x)^{19}\right]_{x^9} \overset{B.T.}{=} \left[\sum_{i=0}^{19}\binom{19}{i}2^i(-x)^{19-i}\right]_{x^9} = \binom{19}{10}2^{10}(-1)^9 = -94595072.$$

b)

$$\left[(x+1/x)^{100}\right]_{x^k} \overset{B.T.}{=} \left[\sum_{i=0}^{100}\binom{100}{i}x^i(x^{-1})^{100-i}\right]_{x^k} = \left[\sum_{i=0}^{100}\binom{100}{i}x^{2i-100}\right]_{x^k}$$

Thus, if k is odd, the coefficient is 0. If $k < 100$ or $k > 100$ it is also 0. Otherwise, the coefficient is $\binom{100}{\frac{k+100}{2}}$.

c)

$$\left[(x-2/x+y)^{100}\right]_{x^{50}y^{40}} \overset{B.T.}{=} \left[\sum_{i=0}^{100}\binom{100}{i}(x-2/x)^i y^{100-i}\right]_{x^{50}y^{40}} \overset{i=60}{=} \left[y^{40}\binom{100}{60}(x-2/x)^{60}\right]_{x^{50}y^{40}}$$

$$\overset{B.T.}{=} \left[y^{40}\binom{100}{60}\sum_{i=0}^{60}\binom{60}{i}x^i(2x^{-1})^{60-i}\right]_{x^{50}y^{40}} = \left[y^{40}\binom{100}{60}\sum_{i=0}^{60}\binom{60}{i}2^{60-i}x^{2i-60}\right]_{x^{50}y^{40}} \overset{i=55}{=} \binom{100}{60}\binom{60}{55}2^5$$

$$= 2402407127747577748733112157894932480$$

Okay, maybe in this case, no need to compute it explicitly.

Problem 10

Find the number of lattice paths from (0,0) to (10,10) (using only "East" and "North" steps, and of minimal length, i.e. 20) not passing through (2,4), nor (4,6), nor (6,9).