Exercise Sheet 1 (Partial Solutions)

Discrete Mathematics I - SoSe17

Problem 2

a. Give an example of a function $f : A \to A$ such that $f^2 = f \circ f = f$ and f is not the identity function.

$f : \{1, 2\} \to \{1, 2\}, f(1) = 1, f(2) = 1.$

b. Prove that if a function $f : A \to A$ is not the identity function and $f^2 = f$, then f is not invertible.

There exists $a \neq b$ such that $f(b) = a$, so that $f^2(b) = f(f(b)) = f(a) = a$ and f is not injective, thus not invertible.

c. Give an example of an invertible function $f : A \to A$, such that $f^3 = f$, yet $f^2 = f \circ f \neq f$.

$f : \{1, 2\} \to \{1, 2\}, f(1) = 2, f(2) = 1.$

d. Give an example of a noninvertible function $f : A \to A$, such that $f^3 = f$, yet $f^2 = f \circ f \neq f$.

$f : \{1, 2, 3\} \to \{1, 2, 3\}, f(1) = 2, f(2) = 1, f(3) = 1.$

Problem 3

Prove that the set of odd natural numbers is infinite.

Let O be the set of odd natural numbers. Assume it is equinumerous to a set of the form $[n]$ for some $n \in \mathbb{N}$. By the Well-Ordering Axiom, O has a least member ℓ. Consider $O' := \{o + 2 : o \in O\}$. Clearly $\ell \notin O'$. But O' is equinumerous to O and consists of odd integers. Hence the union $O \cup O'$ consists of at least $n + 1$ odd integers, a contradiction.□

Problem 4

Which is larger $99^{50} + 100^{50}$ or 101^{50}? Why?

By the Binomial Theorem:

$$A := (100 + 1)^{50} = \sum_{i=0}^{50} \binom{50}{i} 100^i = 100^{50} + \sum_{i=0}^{49} \binom{50}{i} 100^i$$

and

$$B := (100 - 1)^{50} + 100^{50} = 100^{50} + \sum_{i=0}^{50} \binom{50}{i} 100^i (-1)^{50-i}.$$

Now, $A - B$ is

$$-100^{50} + \sum_{i=0}^{49} \binom{50}{i} 100^i \left[1 - (-1)^{50-i}\right] = -100^{50} + 2\sum_{i=0}^{25} \binom{50}{2i-1} 100^{2i-1}.$$

Rearranging gives

$$\underbrace{-100^{50} + 2 \cdot 50 \cdot 100^{49}}_{=0} + \underbrace{2\sum_{i=0}^{24} \binom{50}{2i-1} 100^{2i-1}}_{>0} > 0.$$

Hence 101^{50} is greater than $99^{50} + 100^{50}$.

Problem 5

A Berlin pizzeria advertises that they offer over a million possible pizzas. How many different toppings must they offer if their advertisement is true?

Let T be the set of toppings at the pizzeria. Each possible pizza corresponds to a unique subset of toppings in T. Therefore, the pizzeria has $2^{|T|}$ possible pizzas. By calculation, $2^{19} = 524288$ and twice that value is greater than 1 million, hence $|T| \geq 20$.

Problem 6

A *word* is a concatenation of letters. A *palindrome* is a word that reads the same forward and backwards (for example, *civic* and *radar*). Find the number of n letter words that **are not** palindromes. (Hint: consider two cases depending on the parity of n).

Assume n is even. We count the number of palindromes and subtract it from 26^n, the total number of words of length n. To get a palindrome, there are $n/2$ letters to choose and automatically the remaining letters are fixed, i.e. $26^{n/2}$ palindromes on n letters. Therefore there are $26^{n/2}(26^{n/2} - 1)$ words that are not palindromes on n letters.

Assume n is odd. We give a slightly different counting argument. To get a word, fix the first $(n+1)/2$ letters; there are $26^{(n+1)/2}$ choices, by the M.P.. Then, there is a unique word completion that gives a palindrome. By removing this choice, we get $26^{(n+1)/2}(26^{(n-1)/2} - 1)$ words that are not palindromes.

Problem 7

Find the number of 7 letter words that end with an "a" or do not contain the letter "a".

Problem 8

Let $n \geq 3$. A group of n people, including Alice, Bob, and Carl, is to be seated around a table. Bob refuses to sit next to either Alice or Carl. Find the number of configurations that respect Bob's restriction.

Problem 9

1. How many different words can be obtained by permuting the letters of the word humuhumunukunukuapuaa?

2. How many license plates of the form *ABC 123*, i.e. three letters followed by three numbers, are there?

3. In how many ways can we select a black square and a white square on a chessboard in such a way that the two squares are not in the same column or the same row?

Problem 10

Assume that $S \subseteq 2^{[8]}$ is such that each subset in S has cardinality 4 and each element of [8] belongs to exactly 3 subsets in S. How many subsets are there in S? Write down such an S.

Every element in [8] is contained in 3 subsets; hence there are 24 pairs (a, b) where $a \in [8]$ and b is a subset of [8] that contains a. Since all subsets in S contain 4 elements, we need 6 subsets to get 24 pairs. To get an example of S, label the 8 vertices of a cube from 1 to 8 and form the subsets that correspond to the 6 square faces.