Exercise Sheet 11

Discrete Mathematics I - SoSe17

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Due date 5 July 2017 -- 16:00

You should solve all of the exercises below, and select three to four solutions to be submitted and graded. We encourage you to submit in pairs, please remember to

i) indicate the author of **each** individual solution,
ii) the **name of both team members** on the cover sheet,
iii) **read carefully** the question.
iv) **drafts** are not evaluated and worth **0 pt**.

Problem 1

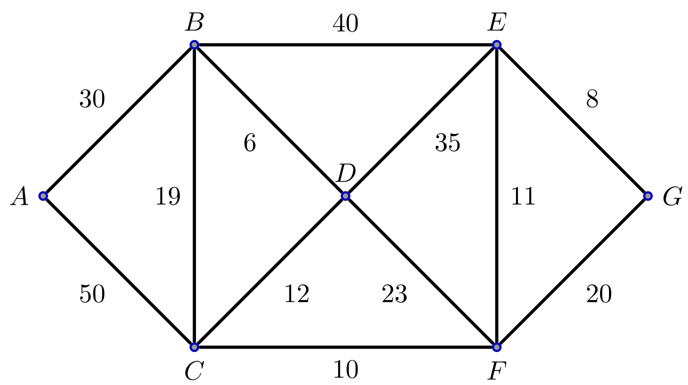
Prove that a simple graph G and its complement $\overline{G}$ (the graph on the same vertex set, with edges given by non-neighbor vertices in G) cannot both be disconnected.

Problem 2

Let $G = (V, E)$ be a Hamiltonian graph and S be any set of k vertices in G. Prove that the graph $(V \setminus S, E')$, where $E' := E \cap (V \setminus S)^2$, has at most k components.

Problem 3

Use Dijkstra's algorithm to determine all shortest paths starting at vertices A, D and G in the following graphs.



Problem 4

Find the minimal spanning tree using Kruskal's algorithm for the graph of Problem 3.

Problem 5

Prove that if T is a spanning tree of a graph G, then for every edge e of G not in T, there exists an edge e' of T such that $T \setminus \{e'\} \cup \{e\}$ is a spanning tree of G again.

Problem 6

What is the number of spanning trees of $K_{m,n}$? (Prove!)

Problem 7

Let $n \geq 1$. Create a graph G with vertex set $V := \{0,1\} \times [n]$, with edges between two vertices $v, w \in V$ whenever they differ at exactly one entry. What is the number of spanning trees of G? (Use generating functions! ... don't be afraid!)

Problem 8

Show that every automorphism of a tree leaves at least one vertex or one edge fixed.

Problem 9

For n math students, each of whom drives their own bike to the university, there are n bike places available in front of the math building. Each student has a prefered spot. In fact, student i prefers $g(i)$, $1 \leq g(i) \leq n$. The students arrive at the university one after the other (not all before lecture starts): first 1, second 2, and so on. The i-th student parks her/his bike in space $g(i)$ if it is free. Otherwise, she or he takes the next number $k > g(i)$ that is available. Otherwise, if no such space is free, the student goes home and never comes back.

Let $P(n)$ be the number of functions g that allow each student to park the bike. Determine $P(n)$. (Hint: $P(2) = 3, P(3) = 16, P(4) = 125$).

Examples: If $g(1) = 3, g(2) = g(3) = 2, g(4) = 1$, then 1 parks in 3, 2 parks in 2, 3 parks in 4, and 4 parks in 1. If $g(1) = 2, g(2) = g(3) = 3, g(4) = 2$, then 1 parks in 2, 2 parks in 3, 3 parks in 4, 4 quits university!

Problem 10

Call a graph *cubic* if each vertex has degree 3. The complete graph K_4 is the smallest example of a cubic graph. Find an example of a connected cubic graph that does not have a Hamiltonian chain.