Formative Midterm Exam - Solution sketchs

Discrete Mathematics I - SoSe17

Problem 1

Let $n \geq 0$. Give a combinatorial proof of the following equality

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

[/10 pts]

Proof. We prove the equality by counting in two different ways how many possibilities there are to form a committee with a president and possibly other members from $n \geq 0$ people.

First if $n = 0$, there is no way that we can get a committee with at least one president in it. Both sides of the equality give zero in this case, which proves this case.

Assume $n \geq 1$. The RHS counts the number of possibilities in two consecutive steps: 1) select the president (n choices), and then 2) select the other members, that is any subset of an $(n-1)$-element set (2^{n-1} choices). Hence by the M.P. there are $n2^{n-1}$ possibilities. The LHS counts the number of possibilities in $n+1$ *disjoint cases* and then two steps. The cases depend on the size k of the committee at the end and the two steps are: 1) Select k committee members (with $0 \leq k \leq n$) and then 2) elect the president. By the M.P. in each each there are $k\binom{n}{k}$ possibilities; further, by the A.P. there are $\sum_{k=0}^{n} k\binom{n}{k}$ ways to fulfill the requirements of the committee.

Since we counted the same number in two different ways, the two numbers should be equal giving the desired equality.□

Problem 2

Let $m \geq 1$ and $a_i \in \mathbb{N} \setminus \{0\}$, for all $i \in [m]$. Show that there exist integer numbers j, k with $0 \leq j < k \leq m$ such that $\sum_{i=j+1}^{k} a_i$ is divisible by m.

[/10 pts]

Let $S_\ell := \sum_{i=1}^{\ell} a_i$ for $\ell \in [m]$.

If there is a $\ell \in [m]$ such that m divides S_ℓ the proof is done.

Otherwise, consider the m numbers S_ℓ, with $\ell \in [m]$ modulo m. Since they are not congruent to $0 \mod m$, by the Pigeonhole Principle, there exist two indices $0 < j < k \leq m$ such that $S_k - S_j \cong 0 \mod m$. In other words, m divides $S_k - S_j = \sum_{i=j+1}^{k} a_i$.

Problem 3

Give the number of integer solutions to the following (in)equalities.

a) $x_1 + x_2 + x_3 \leq 6$, where $x_1, x_2, x_3 \geq 0$.

b) $x_1 + 2x_2 + 5x_3 = 22$, where $x_1, x_2, x_3 \geq 1$.

[/15 pts]

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a) Introduce a fourth variable x_4 (called a slack variable) to make sure that the sum is 6.

We have to place 6 undistinguishable units into 4 labeled variables. This can be achieved by permuting six 1's and three separators "|". To get a permutation, it suffices to choose the locations of the separators; this is done in $\binom{9}{3} = 84$ ways.

b) Make a change of variables $y_i = x_i - 1$, for $i \in \{1, 2, 3\}$ to obtain the system $y_1 + 2y_2 + 5y_3 = 14$ such that $y_i \geq 0$.

This is equal to the number of ways to partition the number 14 into block of possible size 1,2 or 5. There can be zero, one or two blocks of size 5. We enumerate all possibilities:

$$\begin{aligned}14 &= 5+5+2+2 = 5+5+2+1+1 = 5+5+1+1+1+1 \\ &= 5+2+2+2+2+1 = 5+3\cdot 2+3\cdot 1 = 5+2\cdot 2+5\cdot 1 = 5+2+7\cdot 1 = 5+9\cdot 1 \\ &= 7\cdot 2 = 6\cdot 2+2\cdot 1 = 5\cdot 2+4\cdot 1 = 4\cdot 2+6\cdot 1 = 3\cdot 2+8\cdot 1 = 2\cdot 2+10\cdot 1 = 2+12\cdot 1 = 14\cdot 1.\end{aligned}$$

Thus, there are 16 possibilities.

Problem 4

Solve the following recurrence relations.

a) $R_n = R_{n-1} + 2R_{n-2} + (-1)^n$, $(n \geq 2)$, where $R_0 = R_1 = 1$.

[/20 pts]

This is a non-homogeneous linear recurrence relation with constant coefficient.

The characteristic polynomial is $\chi(R) = x^2 - x - 2 = (x+1)(x-2)$. By the Char. Polynomial method, $\alpha 2^n + \beta(-1)^n$, with $\alpha, \beta \in \mathbb{R}$, is the general solution of the homogeneous system.

Since -1 is a multiplicity 1 root of $\chi(R)$ and $g(n) = (-1)^n$, a particular solution of the non-homogeneous system has the form $\gamma n(-1)^n$, for some $\gamma \in \mathbb{R}$.

Therefore $\alpha 2^n + \beta(-1)^n + \gamma n(-1)^n$ is the general solution to the non-homogeneous system (by the Superposition Theorem seen in class).

In order to get the solution, we compute one more value: $R_2 = 4$.

We solve the system

$$\alpha + \beta = 1 = R_0$$
$$2\alpha - \beta - \gamma = 1 = R_1$$
$$4\alpha + \beta + 2\gamma = 4 = R_2$$

To get $\alpha = \frac{7}{9}, \beta = \frac{2}{9}$, and $\gamma = \frac{1}{3}$. Thus the solution is $R_n = \frac{7}{9}2^n + \frac{2}{9}(-1)^n + \frac{n}{3}(-1)^n$, for $n \geq 0$.

Problem 5

Let $n \in \mathbb{N}$. Give a formula for the number of strings consisting of 0,1, or 2's of length n, such that all symbol appears an odd number of times.

[/15 pts]

We use generating functions.

Since the objects we are interested in have an inherent ordering, we have to use exponential generating functions. (A string comes with an order on letters.)

The exp. generating function $Z^e(x)$ for strings of odd length consisting of only zeros is

$$Z^e(x) = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots,$$

because there is only 1 string possible for each length. Similarly, for the exp. generating functions for odd strings of 1's $O^e(x)$ and odd string of 2's $T^e(x)$, so that we get $Z^e(x) = O^e(x) = T^e(x) = \frac{e^x - e^{-x}}{2}$.

In order to create all strings with an odd number of 0's, 1's and 2's, we have to multiply the three generating functions of the odd strings of 0's, 1's and 2's. By multiplying them, we count all the possible ways to arrange an odd number of 0's, 1's and 2's in order to form a bigger string.

So

$$Z^e(x) \cdot O^e(x) \cdot T^e(x) = \left(\frac{e^x - e^{-x}}{2}\right)^3 = \frac{1}{8}(e^{3x} - 3e^x + 3e^{-x} - e^{-3x}).$$

We rewrite as an infinite sum:

$$Z^e(x) \cdot O^e(x) \cdot T^e(x) = \frac{1}{8}\sum_{n=0}^{\infty}(3^n - 3 + 3(-1)^n - (-3)^n)\frac{x^n}{n!}.$$

Hence the coefficient σ_n of x^n which represents the number of strings with odd number of 0's, 1's and 2's is

$$\sigma_n = \begin{cases} 0 \text{ if } n \text{ is even,} \\ \frac{3}{4}(3^{n-1} - 1) \text{ if } n \text{ is odd.} \end{cases}$$

Problem 6

Prove that every finite join-semilattice with $\hat{0}$ is a lattice.

[/15 pts]

Proof. Let L be a finite join-semilattice with $\widehat{0}$. Consider two distinct elements x and y in L, and their principal lower ideals $\langle x \rangle$ and $\langle y \rangle$. Let $F := \langle x \rangle \cap \langle y \rangle$. Since $\widehat{0} \in L$ and $|L| < \infty$, we have $0 < |F| < \infty$.

The subset F is a finite poset induced by L, hence it has some maximal element m.

Claim: m is the only maximal element in F.

Proof of claim: Let m_1 and m_2 be two maximal elements in F. Since L is a join-semilattice $m_1 \vee m_2$ *exists*. Furthermore since $m_1, m_2 \leq x$ and $m_1, m_2 \leq y$ and by the definition of join, we get that $m_1 \vee m_2$ has to be smaller than x AND y. This implies that $m_1 \vee m_2 \in F$. Since m_1 and m_2 are maximal in F the join $m_1 \vee m_2$ has to be equal to m_1 or m_2, hence they shoul be equal.

Since elements in F are lower bounds for $\{x, y\}$ and m is the greatest by the claim, we have that m is the meet of x and y. Thus L is a lattice.□

Problem 7

True or False?

a) $\binom{r}{k} = (-1)^k \binom{k-r-1}{k}, \quad \forall k \in \mathbb{Z}.$

b) Let $f : P \to P$ be an order preserving bijection and $|P| < \infty$. The inverse f^{-1} is order preserving.

c) 101^{50} is smaller than $99^{50} + 100^{50}$.

d) $1^3 + 2^3 + \cdots + n^3 = \binom{n}{2}^2$.

e) Let $S := \{\varnothing, a, b, aa, ab, ba, bb, aba, abb, aab, bab, abab\}$ and define $s \lhd t \quad \Leftrightarrow$ s is a subsequence of t. The poset $(S, \lhd)$ is a lattice.

[/(3x5) pts, -2pt per wrong answer, 1pt for no answer, minimum: 0pts]

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a)	True
b)	True
c)	False
d)	False
e)	False