Final Exam - Take 2

Freie Universität Berlin
Institut für Mathematik
Discrete Mathematics 1

Sommer Semester 2017
11th October 2017

First Name: ______________________________

Last Name: ______________________________

Matricule #: ______________________________

Would you like your results listed online with the last four digits of your matricule number?

Yes ☐ No ☐

Follow the instructions below.

1) You should solve all the exercises below **without** usage of any reference material.
2) Provide **complete** justifications of solutions and state precisely any theorems you use from the lectures.
3) The solutions to the problems should be written **directly on this document**. If necessary, you can join A4 blank sheets for longer solutions **with your name on the top right corner** (else the sheet is considered void).
4) Drafts and sketches of solutions should be written on separate sheets and **not submitted**; only final solutions should be submitted.
5) Arguments that should not be evaluated should be crossed-out with an "X" or be strikethrough once.

Evaluation

Problem 1		15
Problem 2		15
Problem 3		20
Problem 4		15
Problem 5		15
Problem 6		20
Total		100

Problem 1

Give (and justify!) in how many ways it is possible to distribute

a) 6 undistinguishable objects in 3 undistinguishable boxes.

b) 6 undistinguishable objects in 3 labeled boxes.

c) 6 labeled objects in 3 undistinguishable boxes.

d) 6 labeled objects in 3 labeled boxes.

[/15 pts]

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a) This is equal to the number of partitions of 6 into three possibly empty parts: $6+0+0 = 5+1+0 = 4+2+0 = 3+3+0 = 4+1+1 = 3+2+1 = 2+2+2$. There are 7 ways.

b) This is equal to arranging 6 red balls and 2 separators on a line. We need to choose the location of the 2 separators among 8 positions: $\binom{8}{2} = 28$ ways.

c) Using a), for each partition, we decide which labeled object goes into each part:

- 6+0+0: 1 way
- 5+1+0: 6 ways
- 4+2+0: $\binom{6}{2}$ ways
- 3+3+0: $\frac{1}{2}\binom{6}{3}$ ways
- 4+1+1: $\binom{6}{2}$ ways
- 3+2+1: $\binom{6}{3}\binom{3}{2}$ ways
- 2+2+2: $\frac{1}{6}\binom{6}{2}\binom{4}{2}$ ways

By the A.P. we get $1 + 6 + 15 + 10 + 15 + 60 + 15 = 122$ ways.

d) By the M.P. there are 3^6 ways.

Problem 2

For every $n \geq 1$, give an explicit bijection between the collection of subsets of $[n]$ with even cardinality and the collection subsets of $[n]$ with odd cardinality.

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Case $n \equiv 1 \mod 2$: Let $\Phi : 2^{[n]} \to 2^{[n]}$ be defined as $\Phi(S) = [n] \setminus S$, for all $S \in 2^{[n]}$. If $|S| \equiv 0 \mod 2$, then $|\Phi(S)| \equiv 1 \mod 2$ and if $|S| \equiv 1 \mod 2$, then $|\Phi(S)| \equiv 0 \mod 2$. The function Φ is the complementation operation and every set has a complement and two sets having the same complement are equal, i.e. Φ is a bijection. Therefore using the above observations, restricting Φ to even or odd subsets give the desired bijection.

Case $n \equiv 0 \mod 2$: Let $\Psi : 2^{[n]} \to 2^{[n]}$ be defined as

$$\Psi(S) \mapsto \begin{cases} ([n] \setminus S) \cup \{n\} \text{ if } n \in S \\ ([n] \setminus S) \setminus \{n\} \text{ if } n \notin S \end{cases}$$

Similarly as Φ, Ψ makes even subsets into odd subsets and vice-versa. Further, Ψ sends even subsets containing n to odd subsets that contain n and similarly for subsets not containing n. Observing again that Ψ is a bijection (injectivity and surjectivity are obtained from the complement map again), we get the desired bijection.□

Problem 3

Let $n \in \mathbb{N}$. Consider the following three 2-dimensional vectors $NW := (-1, 1)$, $NE := (1, 1)$ and $J := (0, 2)$. We form walks from the origin $(0, 0)$ to the horizontal line $y = n$ using a succession of the three vectors as steps. Let W_n be the number of distinct walks so obtained. For $n = 0$, we set $W_0 = 1$ for the *empty walk*, and $W_{-1} = 0$.

a) How many distinct walks are there for $n \in \{1, 2, 3, 4\}$?

b) Show that $\lim_{n\to\infty} \frac{W_{n+1}}{W_n} = 1 + \sqrt{2}$.

c) Prove that $\begin{bmatrix} W_n & W_{n-1} \\ W_{n-1} & W_{n-2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n$, for all $n \geq 1$.

d) Finally deduce that $W_n W_{n-2} - W_{n-1}^2 = (-1)^n$, for $n \geq 1$.

[/20 pts]

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a) $W_1 = 2$, $W_2 = 5$, $W_3 = 12$, $W_4 = 29$

b) Using the characteristic polynomial method, we get $\chi(W) = x^2 - 2x - 1 = (x - 1 - \sqrt{2})(x - 1 + \sqrt{2})$. Therefore the general solution is $W_n = \alpha(1 + \sqrt{2})^n + \beta(1 - \sqrt{2})^n$ for some fixed $\alpha, \beta \in \mathbb{R}$. Hence, using the fact that $|1 - \sqrt{2}| < 1$,

$$\lim_{n\to\infty} \frac{W_{n+1}}{W_n} = 1 + \sqrt{2}.$$

c) If $n = 1$, then

$$\begin{bmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^1.$$

which concludes the base case of the induction on n. Now assume the result to hold for $n - 1$. We have

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} W_{n-1} & W_{n-2} \\ W_{n-2} & W_{n-3} \end{bmatrix} = \begin{bmatrix} 2W_{n-1} + W_{n-2} & 2W_{n-2} + W_{n-3} \\ W_{n-1} & W_{n-2} \end{bmatrix}.$$

When $n \geq 2$, walks can finish with either with NW, NE or J, meaning that we get the recursion $W_n = 2W_{n-1} + W_{n-2}$ by the A.P.. Using it in the above, we get

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} W_{n-2} & W_{n-1} \\ W_{n-1} & W_{n-2} \end{bmatrix}.$$

d) The determinant of $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n$ is $(-1)^n$. Therefore, by b) we get that $W_n W_{n-2} - W_{n-1}^2 = (-1)^n$.

Problem 4

Let $\mathcal{B}_n := (2^{[n]}, \subseteq)$ be the poset of subsets of $[n]$ ordered by inclusion.

a) Give the definition of the join and meet operations in a poset. Further, give the definition of a lattice.

b) Show that $\mathcal{B}_n$ is a lattice and express the join and the meet operations in terms of the poset $\mathcal{B}_n$.

c) Let $p \in \mathcal{B}_n$. Show that there exists an element $q \in \mathcal{B}_n$ such that $p \wedge q = \hat{0} = \varnothing$ and $p \vee q = \hat{1} = [n]$.

d) Show that the element q with the property in c) is unique.

e) Let $p^c := q$ denote the above unique element. Show that the map from $\mathcal{B}_n$ to itself defined by $p \mapsto p^c$ is an order reversing morphism of posets.

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a) Let $p, q \in P$, for some poset P. The *join* of p and q is the least upper bound of p and q (if it exists). The *meet* of p and q is the greatest lower bound of p and q (if it exists). A *lattice* is a poset P where meet and join exist for all pair $p, q \in P$.

b) Let $p, q \in \mathcal{B}_n$. The join $p \vee q$ is $p \cup q$: it is an upper bound and any subset containing p and q has to contain $p \cup q$. The meet $p \wedge q$ is $p \cap q$: the intersection is contained in both (hence a lower bound) and any lower bound of p and q is contained in $p \cap q$.

c) Let $q := [n] \setminus p$. Then $p \wedge q = \varnothing$ and $p \vee q = [n]$.

d) Take any other subset $r\mathcal{B}_n$ which is not equal to q. Then either there is an element i of q not in r, in which case $p \vee r$ will not contain that element i so that $p \vee r \neq [n]$. Otherwise, there is an element j of p in r, in which case $p \wedge r$ contains at least j hence cannot be $\varnothing$. Hence q is unique.

e) Let $a, b \in \mathcal{B}_n$ be such that $a \subseteq b$. Then $a^c = ([n] \setminus a) \supseteq ([n] \setminus b) = b^c$. □

Problem 5

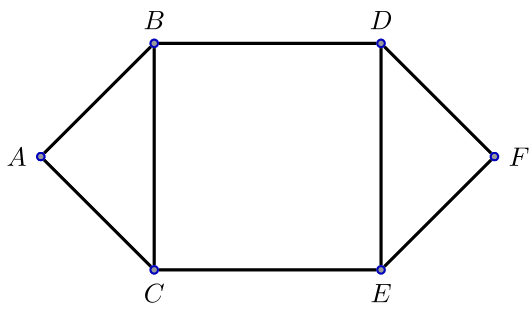
Give the chromatic polynomial of the following graphs:

a) The complete graph K_n, with $n \geq 3$.

b) The null graph N_n, with $n \geq 1$.

c) The cycle graph C_4.

d) The graph G below.



[/15 pts]

Let k be the number of colors.

a) Since all vertices are connected by an edge, no two vertices can have the same color. Start by coloring the vertex 1 (k choices), then color the vertex 2 ($k-1$ choices), etc. up to vertex n ($k-n+1$ choices), by the M.P. we get $\chi(K_n) = \frac{k!}{(k-n)!}$.

b) By the M.P. there are k choices, n times, to color the n vertices. Thus $\chi(N_n) = k^n$.

c) Using the Deletion-Contraction principle: $\chi(\square) = \chi(C) - \chi(\triangle)$, where C is a chain-graph with 4 vertices. Since C is a tree with 4 vertices, we have $\chi(C) = k(k-1)^3$. By a) we get $\chi(\triangle) = k(k-1)(k-2)$. Thus $\chi(\square) = k(k-1)(k^2-3k+3)$.

d) Using c), it remains to choose the colors of A and F. Both have $k-2$ choices or colors. Thus $\chi(G) = k(k-1)(k-2)^2(k^2-3k+3)$.

Problem 6

Let S_n be the number of strings of $0, 1$, and 2's of length $n \geq 0$ with all of the following four properties:

1) an even number of 0's (maybe none),
2) an odd number of 1's,
3) starting and ending with a 2,
4) using at least three 2's.

Give a closed formula for S_n.

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First ignore the first and last 2 appearing in each string. Let $\mathcal{Z}^e(x), \mathcal{O}^e(x)$, and $\mathcal{T}^e(x)$ be the generating functions for

- string of 0's of even length
- string of 1's of odd length
- string of 2's with length at least 1

The strings we need are formed by shuffling one string of 0's, one string of 1's and one string of 2's from the above set, hence multiplying the exponential generating function for each of these family of strings will give the exponential generating function of the desired strings, up to the first and last zero (that we ignore for now).

We have

$$\mathcal{Z}^e(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \cdots = \frac{e^x + e^{-x}}{2},$$

since there is one way to create a string of even cardinality and that the string could be empty. Similarly,

$$\mathcal{O}^e(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \frac{e^x - e^{-x}}{2}.$$

Finally,

$$\mathcal{T}^e(x) = e^x - 1.$$

We multiply the exponential generating functions:

$$\mathcal{Z}^e(x)\mathcal{O}^e(x)\mathcal{T}^e(x) = \frac{e^x - 1}{4}(e^{2x} - e^{-2x}).$$

Writing differently:

$$\frac{e^x - 1}{4}(e^{2x} - e^{-2x}) = \frac{1}{4}(e^{3x} - e^{2x} - e^{-x} + e^{-2x}).$$

The coefficient of x^n in this exponential generating function is

$$\begin{cases} \frac{3^n - 1}{4} & \text{if } n \text{ is even,} \\ \frac{3^n - 2^{n+1} + 1}{4} & \text{if } n \text{ is odd.} \end{cases}$$

This number gives the number of ways to form the strings of 0's, 1's and 2's of length n with the above properties. We now need to add the beginning and ending 2 so that we have at least three 2's and satisfy all conditions. Thus we get:

$$S_n = \begin{cases} \frac{3^{n-2} - 1}{4} & \text{if } n \text{ is even,} \\ \frac{3^{n-2} - 2^{n-1} + 1}{4} & \text{if } n \text{ is odd,} \end{cases}$$

for $n \geq 2$ and $S_1 = S_0 = 0$.