Final Exam

Freie Universität Berlin
Institut für Mathematik
Discrete Mathematics 1

Sommer Semester 2017
26th July 2017

First Name: ____________________

Last Name: ____________________

Matricule #: ____________________

Would you like your results listed online with the last four digits of your matricule number?

Yes ☐ No ☐

Follow the instructions below.

1) You should solve all the exercises below **without** usage of any reference material.
2) Provide **complete** justifications of solutions and state precisely any theorems you use from the lectures.
3) The solutions to the problems should be written **directly on this document**. If necessary, you can join A4 blank sheets for longer solutions **with your name on the top right corner** (else the sheet is considered void).
4) Drafts and sketches of solutions should be written on separate sheets and **not submitted**; only final solutions should be submitted.
5) Arguments that should not be evaluated should be crossed-out with an "X" or be strikethrough once.

Evaluation

Problem 1		15
Problem 2		15
Problem 3		20
Problem 4		15
Problem 5		10
Problem 6		10
Problem 7		15
Total		100

Problem 1

Let $n \geq j \geq 0$. Prove the following equality

$$\sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} = 2^{n-j} \binom{n}{j}.$$

[/15 pts]

Bijective proof: 1) The RHS counts the number of pairs (A, B), where $A \subseteq [n-j]$ and $B \in \binom{[n]}{j}$. There are 2^{n-j} choices for A and $\binom{n}{j}$ choices for B. By the M.P. there are $2^{n-j}\binom{n}{j}$ pairs (A, B).

2) The LHS counts the number of pairs (C, D), where $C \subseteq [n]$, $|C| \geq j$, and $D \subseteq C$ such that $|D| = j$. Partition the pairs depending on the cardinality k of C. For a fixed k, there are $\binom{n}{k}$ choices for C, then there are $\binom{k}{j}$ choices for the set D. By the A.P. and M.P. we have

$$\sum_{k=j}^{n} \binom{n}{k} \binom{k}{j}$$

such pairs.

3) Let $f((A, B)) \mapsto (A \cup B, B)$, for a pair (A, B) as above. Clearly, this map is injective and any pair of type (C, D) above is the image of some pair (A, B). Hence, f is a bijection and both sets of pairs have the same cardinality and the equality follows. $\square$

Algebraic proof: We use the definition of binomial coefficients to get

$$\sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} = \sum_{k=j}^{n} \frac{n!k!}{(n-k)!k!(k-j)!j!}.$$

We can simplify the $k!$ and take out a $\binom{n}{j}$ term:

$$= \sum_{k=j}^{n} \binom{n}{j} \frac{(n-j)!}{(n-k)!(k-j)!}.$$

Taking the binomial coefficient out of the sum and rewriting the summand as a binomial coefficient, we get:

$$= \binom{n}{j} \sum_{k=j}^{n} \binom{n-j}{k-j}.$$

Finally, rewrite the sum by changing the index of the sum, and use the binomial theorem to get:

$$= \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} \overset{B.T.}{=} \binom{n}{j} 2^{n-j}. \ \square$$

Problem 2

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ of $n \geq 1$, its conjugate partition λ^* is the partition obtained by taking the transpose of the Ferrers diagram of λ. A partition is self-conjugate if $\lambda^* = \lambda$. Prove that the number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts.

[/15 pts]

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1) Let S_n be the set of self-conjugate partitions of n. Let O_n be the set of partitions of n into distinct odd parts.

2) The bijection between S_n and O_n is described as follows:

Take a self-conjugate partition λ and consider its Ferrers diagram. We decompose the Ferrers diagram into right-angled hooks. The first hook takes all squares in the first row and first column, the second hooks takes the remaining squares in the second row and second column and so on. The image of λ is then the partition given by the size of the blocks.

3) The size of the i-th part in the image is $2\lambda_i - (2i + 1)$. If two parts have the same size, we get a contradiction with the fact that we started with a partition: the parts will not be decreasing. The size of the i-th part is odd because we started with a self-conjugate partition.

4) This map is clearly injective by construction.

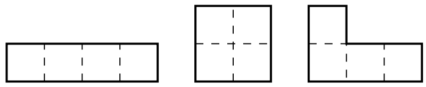
5) This map is surjective: given a partition with distinct odd parts, assemble self-conjugate hooks with the desired sizes and glue them appropriatly to get a Ferrers diagram. Because the parts were self-conjugate, the result will be.

For example, for $\lambda = (5, 4, 3, 2, 1)$, the 15 boxes below are labeled with the index of the part where it is sent: $(9, 5, 1)$ (nine 1's, five 2's, one 3).

1	1	1	1	1
1	2	2	2	
1	2	3		
1	2			
1				

Problem 3

Consider the three tetrominos:



Let $n \geq 0$ and set $T_0 = 1$.

a) Obtain a recurrence relation for the number T_n of tilings of a $2 \times 2n$ rectangle using the above pieces (rotation and reflection allowed).

b) Derive an homogeneous finite order linear recurrence relation with constant coefficients for T_n.

c) Give the appropriate initial conditions for the recurrence relation.

d) Give the **general solution** of this recurrence relation (do not solve it further).

[/20 pts]

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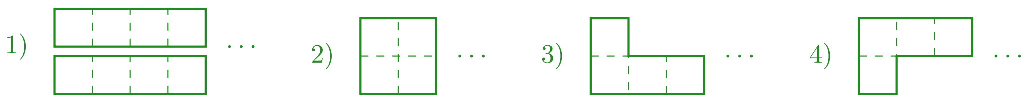
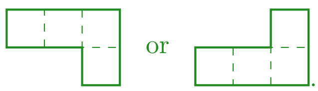
a) Assume we tiled the $2 \times 2n$ rectangle, and consider the first column containing 2 squares. There are four possibilities for which tiles appear in that first column:



In the first two cases, there are exactly T_{n-2} and T_{n-1} possibilities to get the tiling, respectively. For the third case, in order to get a tiling, eventually it has to contain the tetromino



Hence, we partition such tiling depending on the size of the rectangle after the first appearance of one of these two tetrominos (exactly with the above positions). We get $\sum_{i=0}^{n-2} T_i$ such tilings. For the fourth case, it is the same situation, but mirrored. Therefore, the recurrence relation is

$$T_n = T_{n-1} + T_{n-2} + 2\sum_{i=0}^{n-2} T_i.$$

b) Apply the symbolic differenciation method:

$$T_{n+1} = T_n + T_{n-1} + 2\sum_{i=0}^{n-1} T_i, \qquad -(T_n) = -\left(T_{n-1} + T_{n-2} + 2\sum_{i=0}^{n-2} T_i\right),$$

yields

$$T_{n+1} - T_n = T_n - T_{n-2} + 2T_{n-1}.$$

Equivalently, $T_n = 2T_{n-1} + 2T_{n-2} - T_{n-3}$.

c) The initial conditions are $T_0 = 1, T_1 = 1$, and $T_2 = 4$.

d) The characteristic polynomial is $\chi(T) = x^3 - 2x^2 - 2x + 1 = (x+1)(x-a)(x-b)$, where $a = \frac{3+\sqrt{5}}{2}$ and $b = \frac{3-\sqrt{5}}{2}$. Hence $T_n = \alpha(-1)^n + \beta a^n + \gamma b^n$, for some $\alpha, \beta, \gamma \in \mathbb{R}$.

Problem 4

Let P be a finite poset.

a) State the definitions of rank, antichain, minimal element and greatest element of P.

b) Let $\alpha(P)$ be the maximum size of an antichain of P and $\beta(P)$ be the maximum size of a chain of P. Show that $\alpha(P)\beta(P) \geq |P|$.

[/15 pts]

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a)

- The **rank** of a poset is the number of covers in a inclusion maximal chain.
- An **antichain** of a poset P is a subset $A \subseteq P$ such that $\forall x, y \in A$, neither $x \leq y$ nor $y \leq x$ hold.
- A **minimal element** of P is an element $m \in P$ such that there does not exist any $m \neq x \in P$ with the property that $x \leq m$.
- A **greatest element** of P is an element $\hat{1} \in P$ such that $x \leq \hat{1}$, $\forall x \in P$.

b) We show it by induction on the rank of P. If it is zero, then P is an antichain and $\alpha(P) = |P|, \beta(P) = 1$. Therefore $\alpha(P)\beta(P) = |P| \cdot 1 \geq |P|$ holds.

Otherwise, assume it holds for posets of rank $r-1$ and assume that P has rank $r > 0$. Let $M := \{m \in P : m \text{ is minimal element}\}$. Since P is finite M is non-empty. Let $P' := P \setminus M$ be the poset induced by removing the minimal elements. The poset P' is finite and has rank $r-1$. Hence by the induction hypothesis, $\alpha(P')\beta(P') \geq |P'| = |P| - |M|$. Equivalently, $|M| + \alpha(P')\beta(P') \geq |P|$. Since P' is a subposet of P, we have $\alpha(P') \leq \alpha(P)$. Together with the fact that the set M is an antichain by its definition, we get $\alpha(P) + \alpha(P)\beta(P') \geq |P|$. Finally, since $\beta(P) = \beta(P') + 1$ by construction, we get the result.$\square$

Problem 5

a) State Euler's formula for the number of faces (or regions) in a connected simple plane graph.

b) Deduce that every planar simple graph G has vertex v whose degree is at most 5.

[/10 pts]

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a) Let G be a connected plane graph with n vertices and e edges. The number r of regions (or faces) determined by G is $r = e - n + 2$.

b) Let $f_1, f_2, \ldots, f_r$ be the number of edge-curves on the boundary of the r regions of G. By double counting, we get $\sum_{i=1}^{r} f_i = 2e$. Since G is simple, no f_i's is smaller than 3. Therefore $\sum_{i=1}^{r} f_i \geq 3r \iff 2e \geq 3r$. By Euler's formula, we get $2e \geq 3(e-n+2) \iff 2e \geq 3e-3n+6 \iff 3n-6 \geq e$. By the Handshake lemma, we get $\sum_{v \in V} \deg(v) = 2e \leq 6n - 12$. Thus the average $\frac{1}{n}\sum_{v \in V} \deg(v)$ is strictly smaller than 6. That is, there exists a vertex of G whose degree is at most 5. $\square$

Problem 6

Prove that if a graph G has two edge-disjoint spanning trees, then that graph is 2-edge-connected.

[/10 pts]

If G has two edge-disjoint spanning trees, we get the following fact from the properties of trees.

For every $x \neq y \in V$, there are two edge-disjoint path from x to y: 1 per spanning tree.

Therefore, if we remove any 1 edge of E, the graph is still connected. This implies that the edge-connectivity of G is at least 2. Thus G is 2-edge-connected. $\square$

Problem 7

True or False?

a) $x_1+x_2+x_3+x_4=8$, with $x_1,x_2 \geq 2, x_3 \geq 1, 0 \leq x_4 \leq 4$ has 21 nonnegative integral solutions.

b) Every lattice has a greatest element.

c) Every total order with greatest and least element is a well-order.

d) The exponential generating function for the permutations whose cycle notation uses exactly 2 cycles is $\log(1-x)^2$.

e) The graph of the 5-dimensional hypercube is not Hamiltonian but it is Eulerian.

[/(3x5) pts, -2pt per wrong answer, 1pt for no answer, minimum: 0pts]

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a)	False
b)	False
c)	False
d)	False
e)	False