Computation of Cantorian tableaux

Permutation Patterns 2011

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20 June 2011

History

-Brlek, Mendès France, Robson & Rubey, *Cantorian Tableaux and Permanents*, L'Enseignement Mathématique (2004)
-Mendès France, *Cantorian Tableaux revisited*, Funct. Approx. Comment. Math. (2007)
-Brlek, Labbé, Mendès France, *Combinatorial variations on Cantor's diagonal*, (submitted), (2011) preprint available on *arxiv.org*

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s	s^{-1}	s^{-2}	s^{-3}	s^{-4}	s^{-5}	$\cdots$
0	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\cdots$
0	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	$\cdots$
0	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	$\cdots$
0	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\cdots$
0	a_{51}	a_{52}	a_{53}	a_{54}	a_{55}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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0	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\cdots$
0	a_{51}	a_{52}	a_{53}	a_{54}	a_{55}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

We create the number $b = b_1 b_2 b_3 b_4 b_5 \cdots$ where $b_i \neq a_{ii}$

Definitions

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Naturally, we define the permanent of a tableau T

Definition (Brlek et al. (2004))

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$$Perm(T) = \bigcup_{\pi \in S_n} a_{\pi(1)1} a_{\pi(2)2} \cdots a_{\pi(n)n}$$

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Definition (Brlek et al. (2004))

A tableau T is Cantorian if no row-words appear in $Perm(T)$, i.e.

$$L \cap Perm(T) = \varnothing.$$

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Theorem (Brlek et al. (2004))

Let $\mathbb{Q} \subseteq L$ *be a countable set in* $[0,1]$ *and* T *the tableau obtained by the development of the elements in* L *in base* $s \geq 2$. *The tableau* T *is Cantorian. Meaning :*

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If $s = 2$, then we have

$$Perm(T) = [0,1] \setminus L.$$

So, if L contains all algebraic numbers of $[0,1]$, $Perm(T)$ is exactly the set of all transcendental numbers in $[0,1]$.

Examples - finite tableaux

We note the set of $n \times n$ tableaux over the alphabet A containing s letters by $\mathcal{T}_n^s$.

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Sufficient condition

Proposition (Brlek et al. (2004))

If for every row i, there exist another row i' such that $t_{ij} \neq t_{i'j}$, for all j, then the tableau is Cantorian.

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Equivalence relation on tableaux

Remark (Brlek et al. (2004))

The property of being « Cantorian » is invariant :

- *by permuting rows ;*
- *by permuting columns ;*
- *given a bijection of the alphabet, remplacing the elements of a column by their image under this bijection.*

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Equivalence relation on tableaux

Definition

Let $T', T \in \mathcal{T}_n^s$. *We write*

$$T' \sim T \iff T' \text{ can be obtain from } T \text{ by a finite sequence of invariant transformations.}$$

We will say that T' *is equivalent to* T.

Parikh composition

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Definition

Let $w \in A^\star$ be a word of length n. The Parikh composition $\mathfrak{p}_w := \mathfrak{p}(w)$ of w is a composition of weight n and of length s obtain by the function

$$\begin{aligned} \mathfrak{p} : A^\star &\to \mathbb{N}^\star \\ w &\mapsto |w|_{\alpha_1}|w|_{\alpha_2}\cdots|w|_{\alpha_s}. \end{aligned}$$

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We extend the function $\mathfrak{p}$ to tableaux $\mathcal{T}_n^s$, $\mathfrak{P} : \mathcal{T}_n^s \rightarrow (\mathbb{N}^\star)^n$.

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$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

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Orders on $\mathbb{N}^\star$ and $A^\star$

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Finally, we define a total order $\blacktriangleleft$ on $A^\star$, which we call *Parikh composition order* on $A^\star$.

Definition

Let $w, w' \in A^\star$. We write $w \blacktriangleleft w'$ if and only if

$$\mathfrak{p}_w \prec \mathfrak{p}_{w'} \text{ or } \left(\mathfrak{p}_w = \mathfrak{p}_{w'} \text{ and } w \leq w'\right).$$

Total order on tableaux

Definition

Let $T, T' \in \mathcal{T}_n^s$*. We extend naturally the order on tableaux*

$$T \blacktriangleleft T' \iff \begin{array}{l} c_1 \blacktriangleleft c_1' \\ \text{or } (c_1 = c_1', \text{and } c_2 \blacktriangleleft c_2') \\ \text{etc.} \end{array}$$

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Example

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \blacktriangleleft \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \blacktriangleleft \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \blacktriangleleft \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \blacktriangleleft \begin{bmatrix} 2 & 3 & 1 \\ 2 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

Reduced tableaux

Lemma

Let $T \in \mathcal{T}_n^s$. In the class $[T]$, there exist a unique minimal representative $T_\blacktriangleleft$.

Definition

A tableau $T' \in [T]$ is reduced if its Parikh compositions are equal to the ones of $T_\blacktriangleleft$.

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Remark

In general, within a class $[T]$, there are many reduced tableaux.

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Generation of Representatives

Problem

Obtain a algorithm which gives the reduced form of a tableau and which « minimizes » the number of reduced tableaux it outputs within a class (if possible in all classes).

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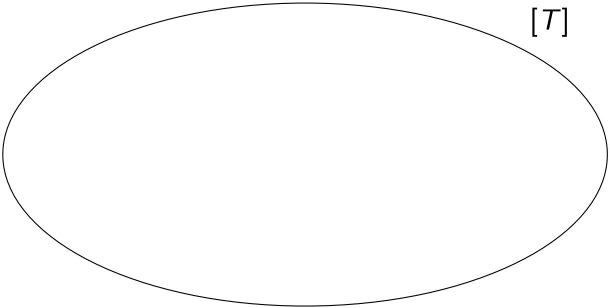

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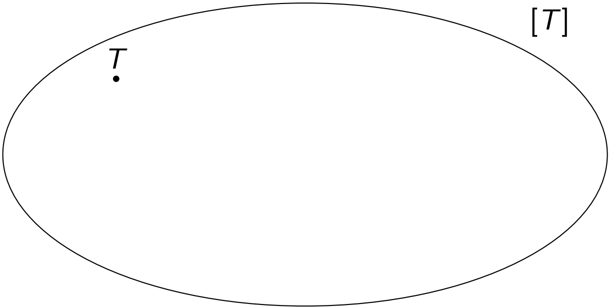

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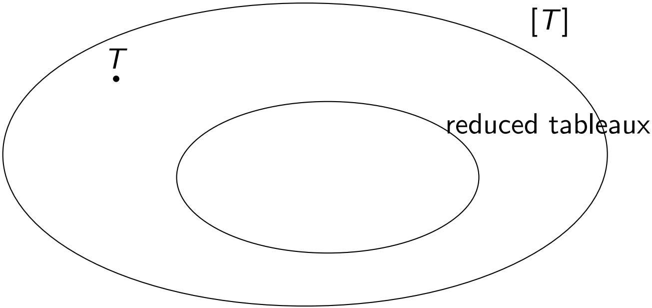

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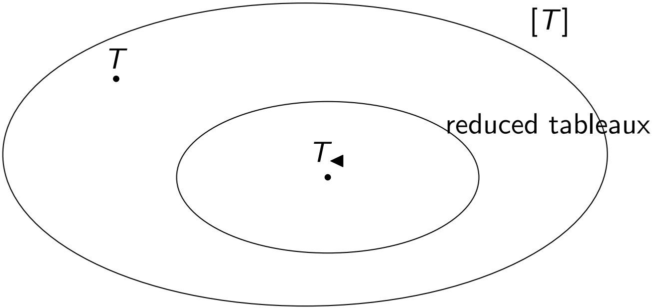

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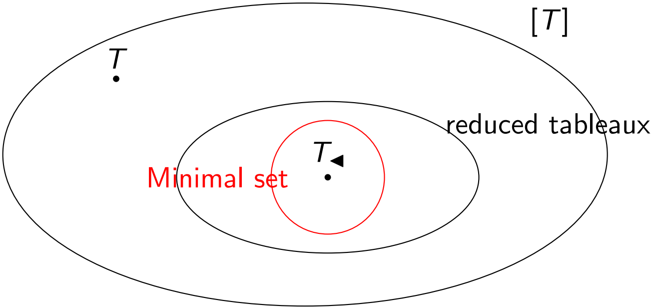

Number of Cantorian classes

Number of minimal representatives of dimension n over an alphabet of s letters (over the number of classes tested) :

$n \backslash s$	2	3	4	5	6
2	1/1	1/1	1/1	1/1	1/1
3	1/3	5/9	5/9	5/9	5/9
4	6/21	56/171	107/275	107/275	107/275
5	11/165	1873/12574			

Cantorian minimal representatives

A few minimal representatives

Cantorian minimal representatives

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Dimension $n = 2$ with $s \geq 2$:

$$\begin{aligned} R_s &= \begin{bmatrix} a & a \\ b & b \end{bmatrix} \\ |[R_s]| &= s^2(s-1)^2 \end{aligned}$$

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The sufficient condition applies.

Cantorian minimal representatives

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Dimension $n = 3$, $s = 2$:

$$
\begin{aligned}
R &= \begin{bmatrix} a & a & a \\ a & a & a \\ b & b & b \end{bmatrix} \\
|[R]| &= 24
\end{aligned}
$$

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Dimension $n = 3$, $s = 3$:

$$\begin{bmatrix} a & a & a \\ a & a & a \\ b & b & b \end{bmatrix} \quad |[R_1]| = 648$$

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$$\begin{bmatrix} a & a & a \\ a & b & b \\ a & c & c \end{bmatrix} \quad |[R_4]| = 324$$

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The sufficient condition does not applies!

Cantorian classes cardinality

Cantorian classes cardinality

Theorem

Let $T \in \mathcal{T}_n^s$. We note the multiplicities of row-words of T by $(f_1, f_2, \ldots, f_q)$, where $q = |L|$. Similarly, $(g_1, g_2, \ldots, g_r)$ denote the multiplicities of column-words of T, where $r = |C|$.

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The cardinality of $[T]$ is

$$\frac{\frac{(n!)^2}{\left(\prod_{i=1}^{r} g_j! \prod_{i=1}^{q} f_i! + \eta\right)} \cdot \prod_{i=1}^{n} \frac{s!}{(s - \ell^{+}(\mathfrak{p}_{c_i}))!}}{\vartheta},$$

where $\vartheta = |\mathcal{O}_{\mathcal{B}}(T) \cap \mathcal{O}_{\Phi}(T)|$ and
$\eta = |\{(\sigma, \tau) \in S_n \times S_n \mid \sigma T \tau = T \text{ and } \sigma T \neq T\}|$.

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$$|[R_4]| = \frac{\frac{(3!)^2}{((2!1!)\cdot(1!1!1!)+0)} \cdot \frac{3!}{(3-1)!} \frac{3!}{(3-3)!} \frac{3!}{(3-3)!}}{6} = 324$$

New enumerative results

Number of Cantorian tableaux of dimension n over an alphabet of s letters :

$n \backslash s$	2	3	4	5
2	$1 \cdot 2^2$	$2^2 \cdot 3^2$	$3^2 \cdot 4^2$	$4^2 \cdot 5^2$
3	$3 \cdot 2^3$	$47 \cdot 2^2 \cdot 3^3$	$207 \cdot 3^2 \cdot 4^3$	$579 \cdot 4^2 \cdot 5^3$
4	$109 \cdot 2^4$	$25036 \cdot 2^2 \cdot 3^4$	$803613 \cdot 3^2 \cdot 4^4$	$9419224 \cdot 4^2 \cdot 5^4$
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	Before	After
$C(4,2)$	3 min.	9 sec.
$C(4,3)$	$\geq$ 17h	105 sec.
$C(4,4)$	$\geq$ 74d	25 sec.

Computed using Sage4.5.2
on a Intel 2.8Ghz machine

New enumerative results

Theorem

The number of Cantorian tableaux $C(n,s)$ for $n = 2, 3$ and 4 is given by the following polynomials

$$
\begin{aligned}
C(2,s) &= s^2 \cdot (s-1)^2; \\
C(3,s) &= s^3 \cdot (s-1)^2 \cdot (s^4 + 2s^3 - 15s^2 + 16s - 1); \\
C(4,s) &= s^4 \cdot (s-1)^2 \cdot (s^{10} + 2s^9 + 3s^8 - 92s^7 - 43s^6 + 1014s^5 \\
&\quad -449s^4 - 5680s^3 + 12045s^2 - 9406s + 2629).
\end{aligned}
$$

Definition

Definition (Brlek et al. (2004))

A tableau T is bi-Cantorian if no row-words or column-words appear in $Perm(T)$, i.e.

$$(C \cup L) \cap Perm(T) = \varnothing.$$

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Fact

The property of being ≪ bi-Cantorian ≫ is not invariant under $\sim$ anymore ! ☹

Brute force enumerative results

$n \backslash s$	2	3	4	5	6	$\cdots$
2	$1 \cdot 2 \cdot 1$	$2 \cdot 3 \cdot 3$	$3 \cdot 4 \cdot 7$	$4 \cdot 5 \cdot 13$	$5 \cdot 6 \cdot 21$	$\cdots$
3	$1 \cdot 2 \cdot 3$	$2 \cdot 3 \cdot 367$	$3 \cdot 4 \cdot 6179$	$4 \cdot 5 \cdot 43065$		
4	$1 \cdot 2 \cdot 91$	$2 \cdot 3 \cdot 402873$				
5	$1 \cdot 2 \cdot 2005$					
6						

TABLE: Number of bi-Cantorian tableaux of dimension n over an alphabet of s letters

Colorings and tableaux

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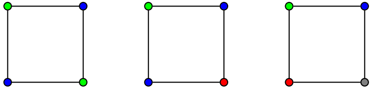
There are three types of coloring :



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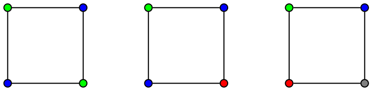

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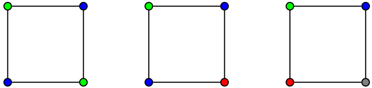

Proposition

The number of bi-Cantorian tableaux $B(s)$ is equalt to the number of coloring $K(s)$. In particular,
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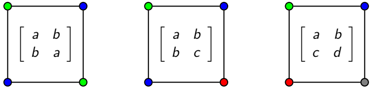

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Problem

Find $\lim_{n\to\infty} \frac{B(n,s)}{C(n,s)}$.
(For $s=2$*, the first values are : 0.5, 0.25, 0.104, 0.045)*

Merci ! Thank you ! Grazie ! Danke ! Gracias !
I am founded by :

Fonds de recherche sur la nature et les technologies Québec
Berlin Mathematical School
LaCIM | Laboratoire de combinatoire et d'informatique mathématique