Universal Oriented Matroids for Subword Complexes of Coxeter Groups

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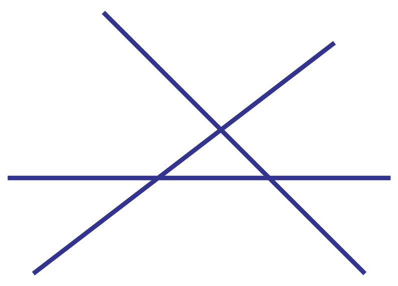
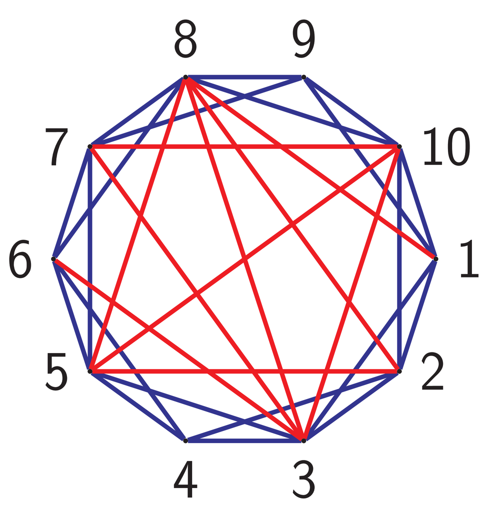
Multi-associahedra (Jonsson 2005)

A k-triangulation of an m-gon is a maximal set of diagonals not containing a $(k+1)$-crossing.



3-crossing



2-triangulation

Definition

The (simplicial) multi-associahedron $\Delta_{m,k}$ is the simplicial complex of $(k+1)$-crossing free sets of k-relevant diagonals of a convex m-gon.

Conjecture

$\Delta_{m,k}$ is the boundary complex of a convex simplicial polytope.

Subword complexes (Knutson–Miller 2004)

Let (W, S) be a finite Coxeter system.
Let $Q = (q_1, \dots, q_r)$ be a word in S, $\pi \in W$.

Definition

The subword complex $\mathcal{SC}(Q, \pi)$ is the simplicial complex for which
faces $\longleftrightarrow$ subsets $I \subset [r]$ such that the subword $Q_{[r]\setminus I}$ contains a reduced expression of π

Subword complexes are homeomorphic to ball or spheres.

Question

Is every spherical subword complex realizable as the boundary complex of convex polytope?

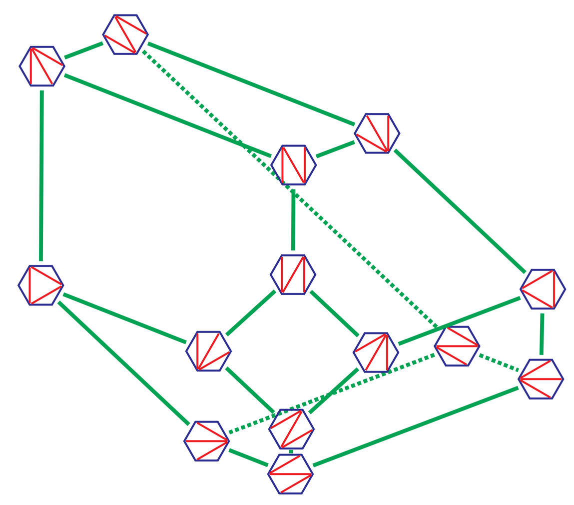
Every multi-associahedron can be obtained as a well chosen subword complex of type A.

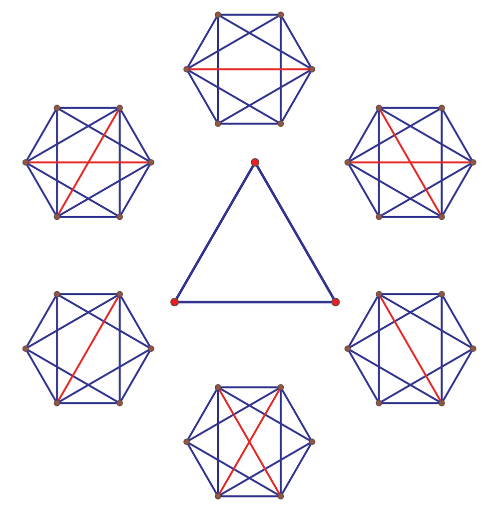
A generalization to finite Coxeter groups including cluster complexes in cluster algebras was found in [3].

Examples

$\Delta_{6,1} = \mathcal{SC}(s_1s_2s_3s_1s_2s_3s_1s_2s_1, [4321])$ is the (simplicial) 3-dim. associahedron.

$\Delta_{6,2} = \mathcal{SC}(s_1s_1s_1, [21])$ is a triangle.

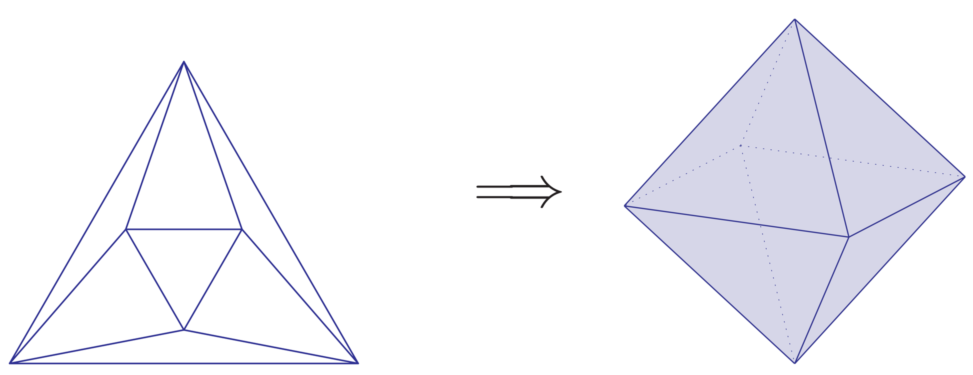




$\Delta_{m,k}$	Realizable as a
$k=1$	dual of a classical associahedron
$m=2k+1$	single vertex
$m=2k+2$	simplex
$m=2k+3$	cyclic polytope
$m=2k+4$	complete simplicial fan [1]
$\Delta_{8,2}$	6-dimensional polytope [2, 1]
$\Delta_{9,2}$	complete simplicial fan [1]
$\Delta_{11,3}$	complete simplicial fan [1]

Steinitz's problem

$\Delta :=$ nice triangulated sphere, i.e. a simplicial sphere.



Question: Is Δ (isom. to) the boundary of a simplicial convex polytope?

Polytopes are easy to get, but are very rare.
vs.
Non-polytopal spheres are very common, but difficult to get.

Underlying Approach

To solve the conjectures it suffices to

1. Obtain complete simplicial fans realizations.
2. Show that they are normal fans of polytopes.

Effective (brute force!) Approach

- Study sign patterns (chirotopes) in the relevant Grassmannian
- $\rightsquigarrow$ Sign functions and parameter matrices

Cool consequences of the approach

- Uses new statistics on reduced words
- Uses Schur functions to describe the realization space

Definition (Punctual Sign Function)

Given 2 reduced words $w = ub_{i,j}v$ and $w' = ub_{j,i}v$ with $\ell(b_{i,j}) = m_{i,j}$.

$$\mathbb{X}(w) := \begin{cases} \mathbb{X}(w') & \text{if } m_{i,j} \equiv 2 \mod 4, \\ -\mathbb{X}(w') & \text{if } m_{i,j} \equiv 0 \mod 4, \\ (-1)^{\kappa+\mu}\mathbb{X}(w') & \text{if } m_{i,j} \equiv 1 \text{ or } 3 \mod 4. \end{cases}$$

where $\kappa = \#$ of letters s_k in u such that $i < k \leq j$ and $\mu = \#$ of letters s_k in v such that $i \leq k < j$.

Determinantal Formulas (à la Binet–Cauchy)

We give a factorization formula for the determinant of matrices

$$\left(f_{i,j}(x_j)\right)_{i,j\in[k]},$$

where $f_{i,j}(x_j)$ is a polynomial in $\mathbb{R}[x_j]$.

Open Problem

Give a general combinatorial interpretation of such matrices.

Theorem (Universality)

Let $Q \in S^r$ and $\mathbf{A} \in \mathbb{R}^{(r-N)\times r}$. If $\mathbf{A}$ is a chirotopal realization of $\mathcal{SC}(Q, w_\circ)$, then there $\exists$ a parameter tensor $\mathcal{P}_{\mathbf{A}}$, and $x_i > 0$, with $i \in [r]$, such that

- *$i < j$ and $q_i = q_j$ implies $x_i < x_j$, and*
- *for every occurrence of each reduced word v of $w_\circ$ which is a subword of Q, the following equality holds*

$$\operatorname{sign}\left(\sum_{\mathfrak{z}\in\mathfrak{Z}_{\alpha_v}} \det[\mathcal{P}_{\mathbf{A}}]_{\mathfrak{z}} \mathcal{S}_{\Lambda_{\mathfrak{z}},\Omega_v}\right) = \mathbb{X}(v),$$

where

$$\mathcal{S}_{\Lambda,P} := \prod_{i=1}^{n} s_{\lambda^i,p_i}.$$

Example

Consider the matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -x_1 & x_2 & -x_3 & x_4 \\ x_1^2 & -x_2^2 & x_3^2 & -x_4^2 \end{pmatrix}.$$

The determinant is

$$\begin{aligned}\det M = &-(x_3-x_1)(x_4-x_2) \\ &\Big[(-1)\left(s_{(0,0),\{1,3\}}s_{(1,0),\{2,4\}}\right) \\ &+(1)\left(s_{(1,0),\{1,3\}}s_{(0,0),\{2,4\}}\right)\Big] \\ = &-(x_3-x_1)(x_4-x_2)(x_1-x_2+x_3-x_4).\end{aligned}$$

Example ($2k$-dimensional cyclic polytope)

Let $W = I_2(4)$, and $\pi = w_\circ$.

- $f_1(x) = (1, 0, -x, x^2)$, $f_2(x) = (0, 1, x, -x^2)$.
- $\forall q_i \in Q$, assign $x_i > 0$ such that $x_j > x_i$ whenever $q_i = q_j$ and $j < i$.
- If $p_i = s_1$, evaluate f_1 at x_i, otherwise $p_i = s_2$ and evaluate f_2 at x_i
- We get 2 conditions.

If $p_{i_1}p_{i_2}p_{i_3}p_{i_4} = s_1s_2s_1s_2$,

$$-1 = \mathbb{X}(s_1s_2s_1s_2) = \operatorname{sign}(x_{i_1} - x_{i_2} + x_{i_3} - x_{i_4}).$$

If $p_{i_1}p_{i_2}p_{i_3}p_{i_4} = s_2s_1s_2s_1$,

$$1 = \mathbb{X}(s_2s_1s_2s_1) = \operatorname{sign}(-x_{i_1} + x_{i_2} - x_{i_3} + x_{i_4}).$$

Equivalently, $x_1 < x_2 < \cdots < x_{2k+3} < x_{2k+4}$.

References

[1] Bergeron, N., Ceballos, C., Labbé, J.-P.: Fan realizations of type A subword complexes and multi-associahedra of rank 3, 2015.

[2] Bokowski, J., Pilaud, V.: On symmetric realizations of the simplicial complex of 3-crossing-free sets of diagonals of the octagon, 2009.

[3] Ceballos, C., Labbé, J.P., Stump, C.: Subword complexes, cluster complexes, and generalized multi-associahedra, 2014.

[4] Jonsson, J., Generalized triangulations and diagonal-free subsets of stack polyominoes, 2005.

[5] Knutson, A., Miller, E.: Subword complexes in Coxeter groups, 2004.

[8] Richard P. Stanley, Enumerative combinatorics. Vol. 2, 1999.

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