Cantorian and bi-Cantorian tableaux

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History

Brlek, Mendès France, Robson & Rubey, *Cantorian Tableaux and Permanents*, L'Enseignement Mathématique (2004)
Mendès France, *Cantorian Tableaux revisited*, Funct. Approx. Comment. Math. (2007)

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0	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	$\cdots$
0	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	$\cdots$
0	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\cdots$
0	a_{51}	a_{52}	a_{53}	a_{54}	a_{55}	$\cdots$
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We create the number $b = b_1 b_2 b_3 b_4 b_5 \cdots$ where $b_i \neq a_{ii}$

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Naturally, we define the permanent of a tableau T

Definition (Brlek et al. (2004))

The permanent of a tableau T is the set of words

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Definition (Brlek et al. (2004))

A tableau T is Cantorian if no row-words appear in Perm(T), i.e.

$$L \cap Perm(T) = \varnothing.$$

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Let $\mathbb{Q} \subseteq L$ *be a countable set in* $[0,1]$ *and* T *the tableau obtained by the development of the elements in* L *in base* $s \geq 2$. *The tableau* T *is Cantorian. Meaning :*

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Theorem (Brlek et al. (2004))

If $s = 2$, then we have

$$Perm(T) = [0,1] \setminus L.$$

So, if L contains all algebraic numbers of $[0,1]$, $Perm(T)$ is exactly the set of all transcendental numbers in $[0,1]$.

Examples - finite tableaux

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Sufficient condition

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Equivalence relation on tableaux

Remark (Brlek et al. (2004))

The property of being « Cantorian » is invariant :

- *by permuting rows ;*
- *by permuting columns ;*
- *given a bijection of the alphabet, remplacing the elements of a column by their image under this bijection.*

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Equivalence relation on tableaux

Definition

Let $T', T \in \mathcal{T}_n^s$. We write

$$T' \sim T \iff T' \text{ can be obtain from } T \text{ by a finite sequence of invariant transformations.}$$

We will say that T' is equivalent to T.

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6. ... in order to study *bi-Cantorian* tableaux.

Parikh composition

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Let $w \in A^\star$ be a word of length n. The Parikh composition $\mathfrak{p}_w := \mathfrak{p}(w)$ of w is a composition of weigth n and of length s obtain by the function

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We extend the function $\mathfrak{p}$ to tableaux $\mathcal{T}_n^s$, $\mathfrak{P} : \mathcal{T}_n^s \rightarrow (\mathbb{N}^\star)^n$.

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Finally, we define a total order $\blacktriangleleft$ on $A^\star$, which we call *Parikh composition order* on $A^\star$.

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Total order on tableaux

A tableau $T \in \mathcal{T}_n^s$ is just a sequence of n words of length n over the alphabet A.

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Definition

Let $T, T' \in \mathcal{T}_n^s$. We define naturally the relation

$$T \blacktriangleleft T' \iff \begin{array}{l} c_1 \blacktriangleleft c_1' \\ \textit{or } (c_1 = c_1', \textit{and } c_2 \blacktriangleleft c_2') \\ \textit{etc.} \end{array}$$

where c_i is the i-th column-word of T and similarly for c_i' with T'.

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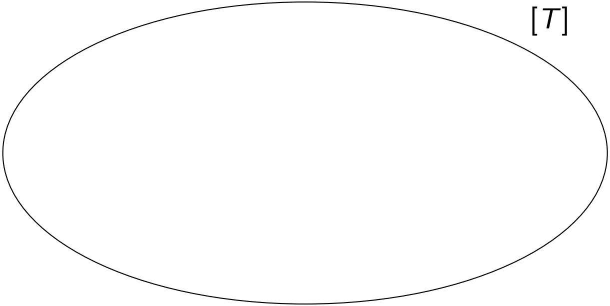

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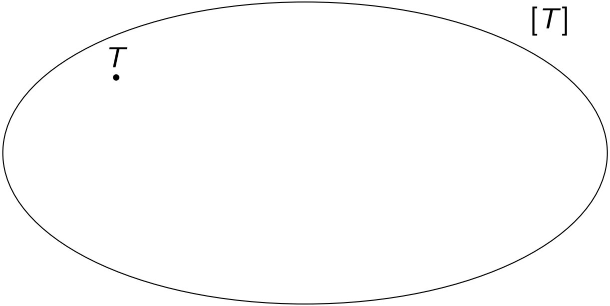

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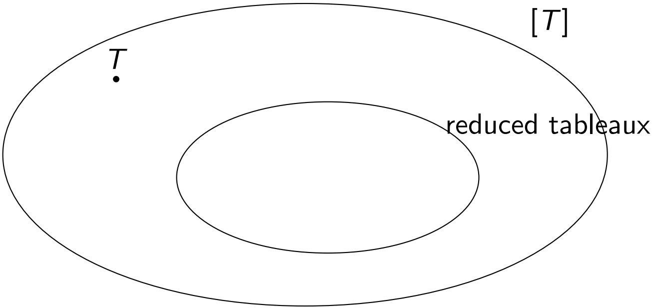

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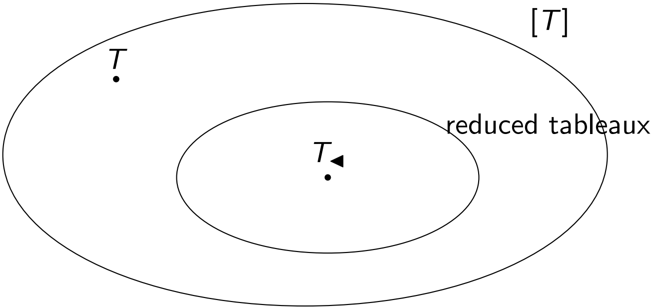

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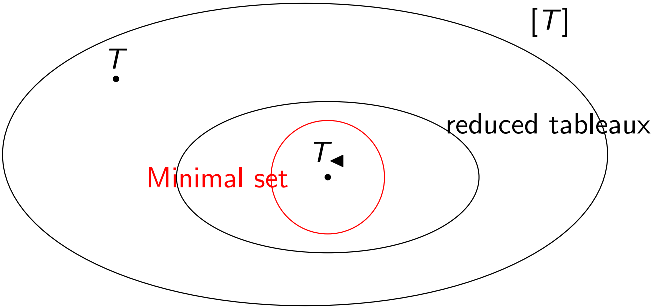

Number of Cantorian classes

Number of minimal representatives of dimension n over an alphabet of s letters :

$n \backslash s$	2	3	4	5
2	1	1	1	1
3	1	5	5	5
4	6	56	105	105
5	11	1875	≥ 10000	≥ 12000

Cantorian minimal representatives

A few minimal representatives

Cantorian minimal representatives

A few minimal representatives
Dimension $n = 2$ with $s \geq 2$:

$$\begin{aligned} R_s &= \begin{bmatrix} a & a \\ b & b \end{bmatrix} \\ |[R_s]| &= s^2(s-1)^2 \end{aligned}$$

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The sufficient condition applies.

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Dimension $n = 3$, $s = 2$:

$$R = \begin{bmatrix} a & a & a \\ a & a & a \\ b & b & b \end{bmatrix}$$

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$$|[R_1]| = 648 \quad |[R_2]| = 1944 \quad |[R_3]| = 1944$$

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The sufficient condition does not applies !

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Let $T \in \mathcal{T}_n^s$. We note the multiplicities of row-words of T by $(f_1, f_2, \ldots, f_q)$, where $q = |L|$. Similarly, $(g_1, g_2, \ldots, g_r)$ denote the multiplicities of column-words of T, where $r = |C|$.

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$$\frac{\frac{(n!)^2}{\left(\prod_{j=1}^{r} g_j! \prod_{i=1}^{q} f_i! + \eta\right)} \cdot \prod_{i=1}^{n} \frac{s!}{(s-\ell^+(\mathfrak{p}_{c_i}))!}}{\vartheta},$$

where $\vartheta = |\mathcal{O}_{\mathcal{B}}(T) \cap \mathcal{O}_{\Phi}(T)|$ and $\eta = |\{(\sigma, \tau) \in S_n \times S_n \mid \sigma T \tau = T \text{ and } \sigma T \neq T\}|$.

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$$|[R_4]| = \frac{\frac{(3!)^2}{((2!1!)\cdot(1!1!1!)+0)} \cdot \frac{3!}{(3-1)!} \frac{3!}{(3-3)!} \frac{3!}{(3-3)!}}{6} = 324$$

New enumerative results

Number of Cantorian tableaux of dimension n over an alphabet of s letters :

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3	$3\cdot 2^3$	$47\cdot 2^2\cdot 3^3$	$207\cdot 3^2\cdot 4^3$	$579\cdot 4^2\cdot 5^3$
4	$109\cdot 2^4$	$6259\cdot 2^4\cdot 3^4$	$748317\cdot 3^2\cdot 4^4$	$8423896\cdot 4^2\cdot 5^4$
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	Before	After
$C(4,2)$	3 min.	9 sec.
$C(4,3)$	$\geq$ 17h	105 sec.
$C(4,4)$	$\geq$ 74j	25 sec.

Computed using Sage4.5.2
on a Intel 2.8Ghz machine

New enumerative results

Theorem

The number of Cantorian tableaux $C(n, s)$ for $n = 2, 3$ and 4 is given by the following polynomials

$$
\begin{aligned}
C(2, s) &= s^2 \cdot (s-1)^2; \\
C(3, s) &= s^3 \cdot (s-1)^2 \cdot (s^4 + 2s^3 - 15s^2 + 16s - 1); \\
C(4, s) &= s^4 \cdot (s-1)^2 \cdot (s^{10} + 2s^9 + 3s^8 - 476s^7 + 4949s^6 \\
&\quad -26250s^5 + 80575s^4 - 146992s^3 + 156429s^2 - 89278s \\
&\quad +21061).
\end{aligned}
$$

Open problems - Updated 2010

In 2008

1. Find a necessary and sufficient condition for a tableau to be Cantorian ; [ ]

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Open problems - Updated 2010

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6. ... in order to study *bi-Cantorian* tableaux.[/]

Open problems - Updated 2010

Some new problems

1. Algorithm to minimize the number of reduced forms ;

Open problems - Updated 2010

Some new problems

1. Algorithm to minimize the number of reduced forms ;
2. Closed formula for the class cardinality.

Definition

Definition (Brlek et al. (2004))

A tableau T is bi-Cantorian if no row-words or column-words appear in Perm(T), i.e.

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Fact

The property of being « bi-Cantorian » is not invariant under $\sim$ anymore ! ☹

Brute force enumerative results

$n \backslash s$	2	3	4	5	6	$\cdots$
2	$1 \cdot 2 \cdot 1$	$2 \cdot 3 \cdot 3$	$3 \cdot 4 \cdot 7$	$4 \cdot 5 \cdot 13$	$5 \cdot 6 \cdot 21$	$\cdots$
3	$1 \cdot 2 \cdot 3$	$2 \cdot 3 \cdot 367$	$3 \cdot 4 \cdot 6179$	$4 \cdot 5 \cdot 43065$		
4	$1 \cdot 2 \cdot 91$	$2 \cdot 3 \cdot 402873$				
5	$1 \cdot 2 \cdot 2005$					
6						

TABLE: Number of bi-Cantorian tableaux of dimension n over an alphabet of s letters

Colorings and tableaux

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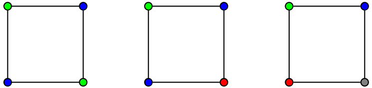

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Let $K(s)$ be the number of colorings of the 4-cycle such that every edge is not monochromatic.
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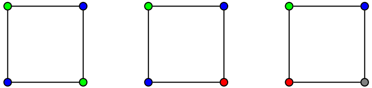

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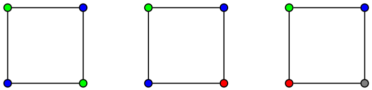

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The number of bi-Cantorian tableaux $B(s)$ is equalt to the number of coloring $K(s)$. In particular,
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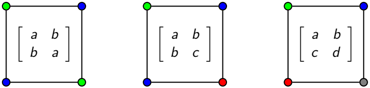

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Problem

Find $\lim_{n\to\infty} \frac{B(n,s)}{C(n,s)}$.

Merci ! Thank you ! Grazie ! Danke ! Gracias !