Polyhedral Combinatorics of Coxeter Groups

Dissertation's Defense

Jean-Philippe Labbé

July 8^{th} 2013

A few motivations

Open Problem (Dyer (1993))

Is there, for each infinite Coxeter group, a complete ortholattice that contains the weak order?

Open Problem (Jonsson (2003))

Is there a polytopal realization of the multi-associahedron?

A few motivations

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Is there, for each infinite Coxeter group, a complete ortholattice that contains the weak order?

$\rightsquigarrow$ Introduction of the limit roots of an infinite Coxeter group.

Open Problem (Jonsson (2003))

Is there a polytopal realization of the multi-associahedron?

$\rightsquigarrow$ Introduction of the multi-cluster complex of a finite Coxeter group.

PART I: Ortholattice for infinite Coxeter groups?
Fixed-gear drivetrain

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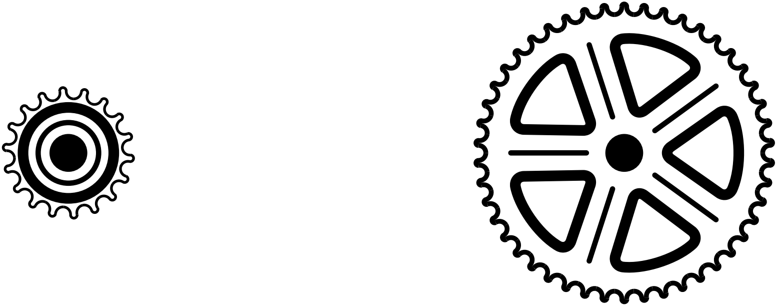

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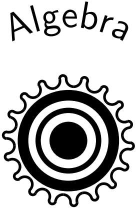

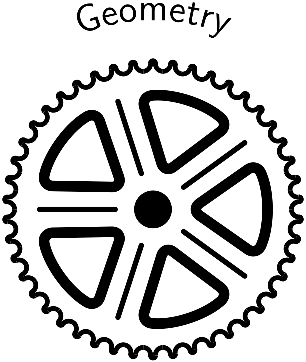

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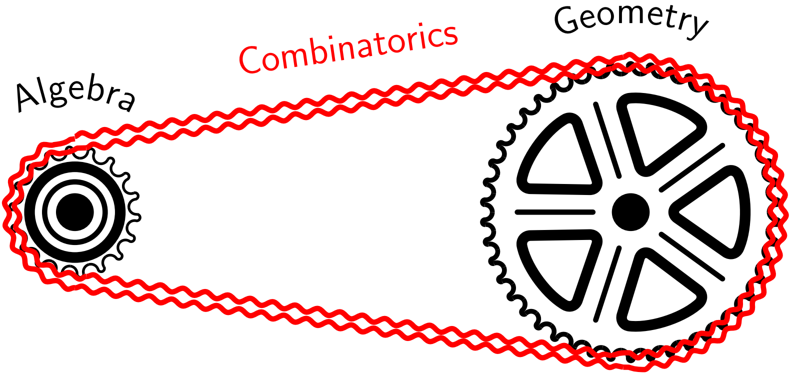

The weak order on Coxeter groups

(W, S) – infinite Coxeter group $\left(\langle s, t \in S | e = s^2 = (st)^{m_{s,t}} \rangle\right)$

Definition (Weak order)

Let $u, v \in W$. Then $u \leq v \iff u$ is a prefix of v.

Example

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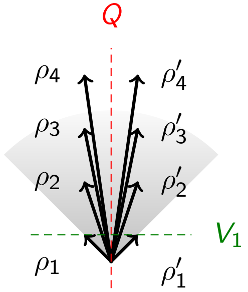
$(W, \leq)$ is a meet-semilattice. There is no maximal element.

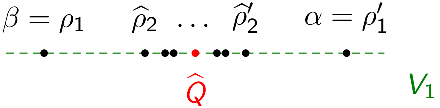
Question

How to tell when the join of two elements exists?

Asymptotical behaviour of roots

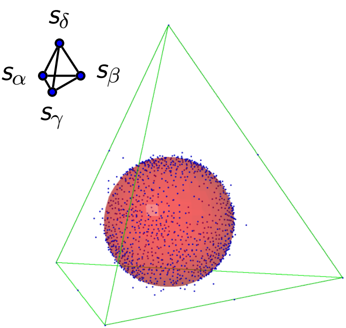
Study the directions of the roots.

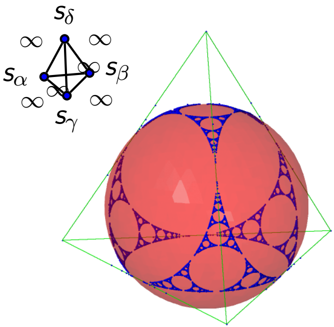




The infinite dihedral group $I_2(\infty)$.

Limit roots of Coxeter groups





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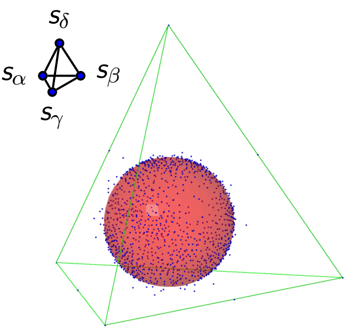

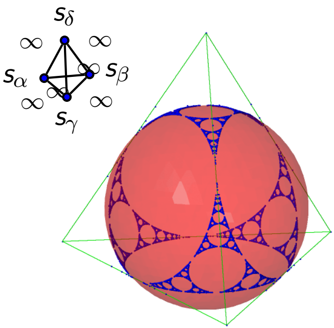

Theorem (Hohlweg-L.-Ripoll, *Canad. J. Math. (2013)*)

The set $E(\Phi)$ of accumulation points of normalized roots $\widehat{\Phi}$ is contained in the isotropic cone of (V, B).

A first step using biconvexity

Theorem (L. 2012)

Let (W, S) be a Coxeter group of rank $n \leq 3$, there is a complete ortholattice containing the weak order.

$\rightsquigarrow$ The join of two elements can be computed using convex hulls.

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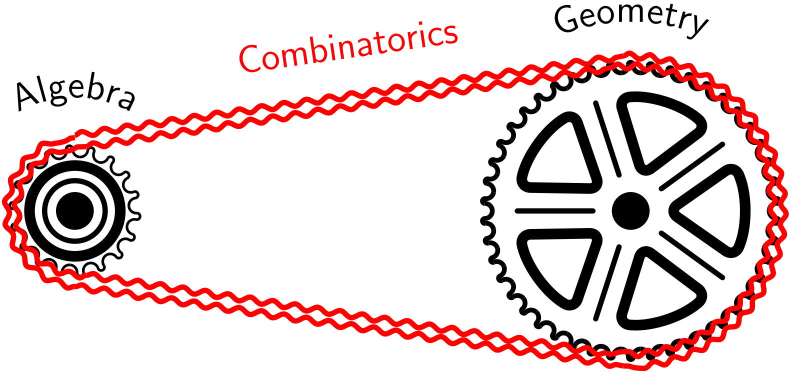
Fact (L. 2012)

For Coxeter groups of rank $n \geq 4$ the join can not be computed using convex hulls.

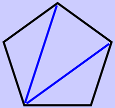
The notion of convexity is to restrictive.
$\rightsquigarrow$ biclosedness seems to be the right geometry to look at.

PART II: Subword Complexes in Discrete Geometry



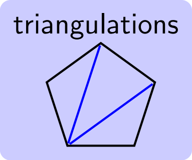
Multi-cluster complexes

triangulations



Multi-cluster complexes

Finite Coxeter Groups (Algebraic)



$\longrightarrow$

Integer $k > 1$
(Geometric)

$\downarrow$

Multi-cluster complexes

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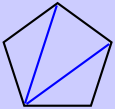

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gen. triangulations

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clusters

Integer $k > 1$
(Geometric)

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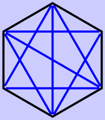
multi-triangulations



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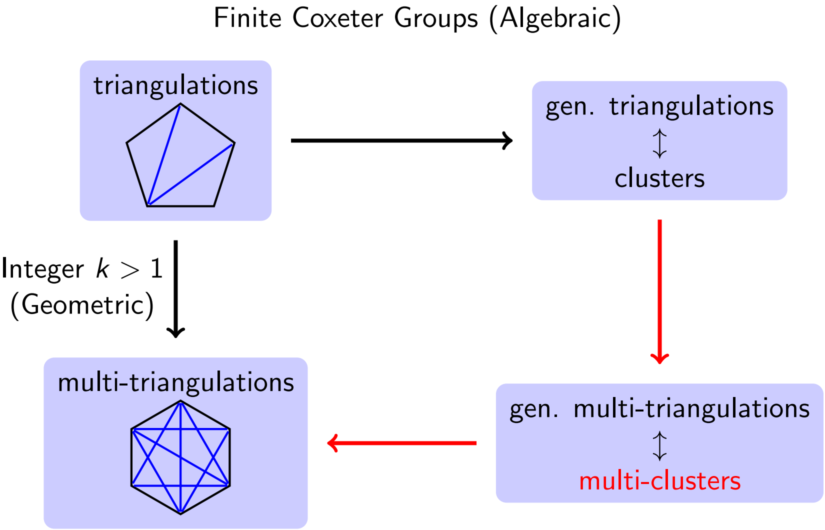

Simplicial complex of multi-triangulations

Fix a convex m-gon.

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$$\text{faces} \quad \longleftrightarrow \quad \begin{array}{c}\text{sets of (relevant) diagonals not containing}\\ k+1 \text{ pairwise crossing diagonals}\end{array}$$

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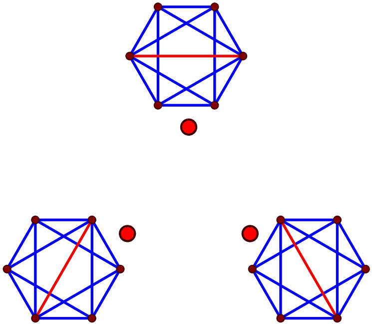
Conjecture (Jonsson, 2003)

$\Delta_{m,k}$ is isomorphic to the boundary complex of a simplicial polytope.

Simplicial complex $\Delta_{m,k}$ - Example

Let $m = 6$ and $k = 2$



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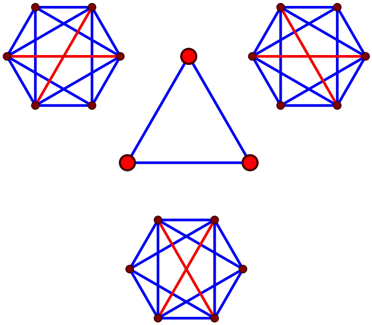

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Subword complexes

(W, S) finite Coxeter system of rank n

Definition (Knutson-Miller, 2004)

Let $Q = (q_1, \ldots, q_r)$ be a word in S and $\pi \in W$.

$$\textit{subword complex } \Delta(Q,\pi) \quad := \quad \begin{array}{c} \textit{simplicial complex} \\ \textit{facets} \longleftrightarrow \textit{complements (in } Q\textit{)} \\ \textit{of reduced expressions of } \pi \end{array}$$

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Theorem (Knutson-Miller, 2004)

Subword complexes are topological spheres or balls.

Multi-cluster complex - Definition

Definition (Ceballos-L.-Stump, *J. of Alg. Comb. 2013*)

The multi-cluster complex $\Delta_c^k(W)$ *is the subword complex* $\Delta(\mathbf{c}^k \mathbf{w}_\circ(\mathbf{c}), w_\circ)$ *of type* W.

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Theorem (CLS, 2013)

The subword complex $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ *is isomorphic to the c-cluster complex of type* W.

Multi-cluster complexes of type A and B

Theorem (Pilaud-Pocchiola 2012, Stump 2011)

The multi-cluster complex $\Delta_c^k(A_n)$ $\cong$ *simplicial complex of k-triangulations of a convex m-gon*

where $m = n + 2k + 1$.

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Theorem (CLS, 2013)

The multi-cluster complex $\Delta_c^k(B_{m-k})$ $\cong$ *simplicial complex of centrally symmetric k-triangulations of a regular convex 2m-gon*

Corollary

$\Delta_{m,k}^{sym}$ *is a vertex-decomposable simplicial sphere.*

Universality and polytopality of $\Delta_c^k(W)$

Question (Knutson-Miller, 2004)

Charaterize all simplicial spheres that can be realized as a subword complex.

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A simplicial sphere is realized as a subword complex $\iff$ *it is the link of a face of a multi-cluster complex $\Delta_c^k(W)$.*

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Corollary

The following two statements are equivalent.

(i) *Every spherical subword complex is polytopal.*

(ii) *Every multi-cluster complex is polytopal.*

Polytopality of multi-cluster complexes

Conjecture ($\Rightarrow$ Knutson-Miller'04, Jonsson'05, Soll-Welker'09)

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- *True for $k = 1$: Chapoton-Fomin-Zelevinsky, Hohlweg-Lange-Thomas, Pilaud-Stump, Stella*
- *True for $I_2(m)$, $k \geq 1$: cyclic polytope, Ceballos-L.-Stump*
- *True for A_3, $k = 2$: Bokowski-Pilaud, Ceballos-L.*

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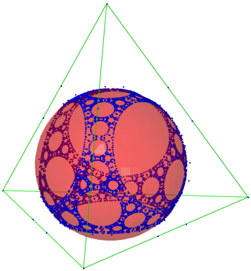
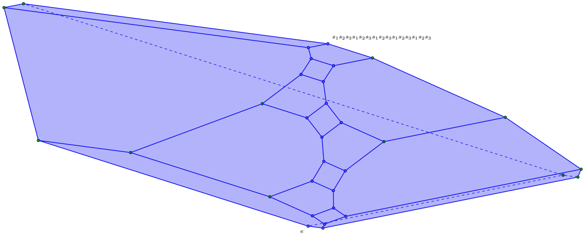
Recent work of Bergeron-Ceballos-L.:

- New combinatorial construction giving 60000+ realizations of A_3, $k = 2$
- Fan realizations of all subword complexes of type A_3

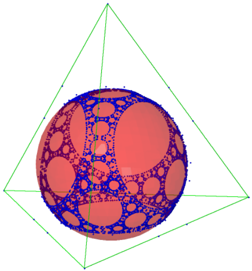
Thanks!



Merci! Thank you! Grazie! Danke! Gracias!



Recent developments on limit roots

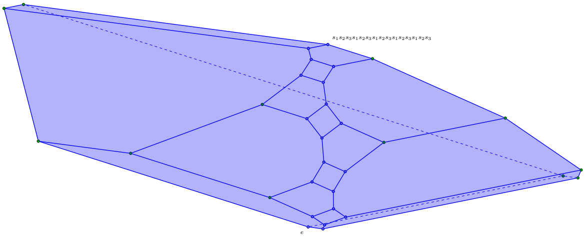


⤳ Relation with the Tits cone [Dyer-Hohlweg-Ripoll, (arxiv:2013)]

⤳ Limit set of Kleinian groups [Hohlweg-Préaux-Ripoll, (arxiv:2013)]

⤳ Sphere packings and geometric invariants for Coxeter groups [Chen-L., (in preparation)]

Recent developments on multi-cluster complexes



$\rightsquigarrow$ Generalized brick polytope, spanning trees [Pilaud-Stump, arxiv:2011-12]

$\rightsquigarrow$ Denominator vectors of cluster algebras of finite types [Ceballos-Pilaud, (arxiv:2013)]

$\rightsquigarrow$ Common vertices of permutahedra and generalized associahedra [L., 2013 and L.-Lange (in preparation)]

Geometric computation of the join