# *Higher-Dimensional Kaleidoscopes*

*Disputation's Talk*

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Freie Universität Berlin

Berlin Mathematical School

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# Definitions and Motivation

### Definition

A kaleidoscope is a cylinder with mirrors containing loose, colored objects such as beads or pebbles and bits of glass.

A Kaleidoscope operates on the principle of multiple reflection, where several mirrors are placed at an angle to one another, usually 60. Typically there are three rectangular mirrors. Setting the mirrors at a 60 so that they form a triangle. As the tube is rotated, the tumbling of the coloured objects presents varying colours and patterns. Arbitrary patterns shows up as a beautiful symmetrical pattern created by the reflections. A two-mirror kaleidoscope yields a pattern or patterns isolated against a solid black background, while the three-mirror (closed triangle) type yields a pattern that fills the entire field.

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In group theory and geometry, a reflection group is a discrete group which is generated by a set of reflections of a finite-dimensional Euclidean space.

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A root system $\Phi$ is a finite set of non-zero vectors of $V$ satisfying the conditions:

(R1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$;

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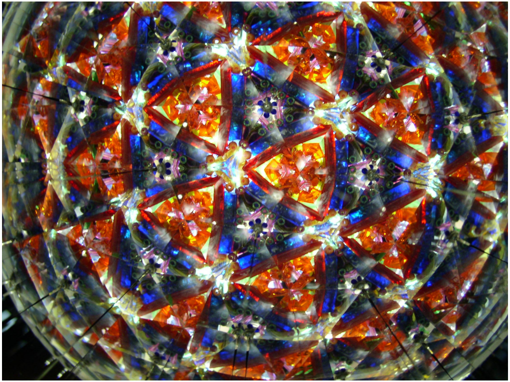
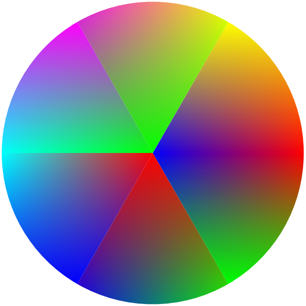
# Kaleidoscopes



from `specialtyglassworks.com`

# How does a Kaleidoscope work?



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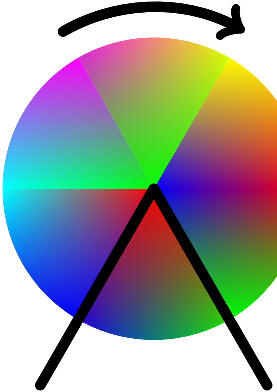

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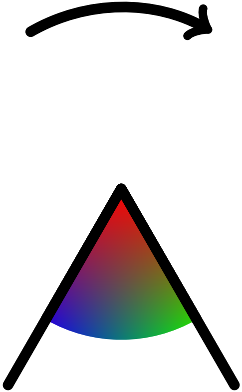

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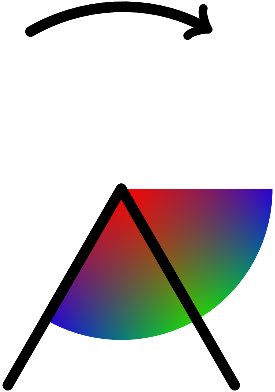

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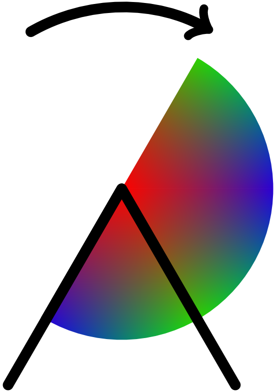

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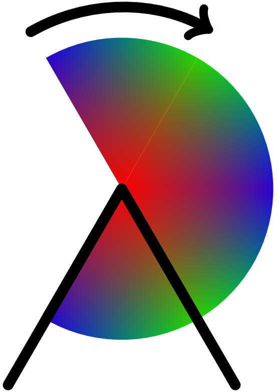

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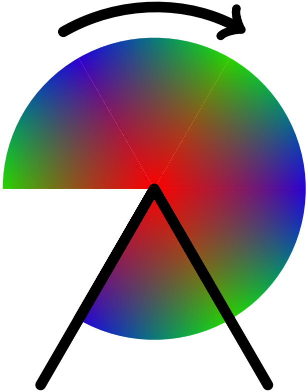

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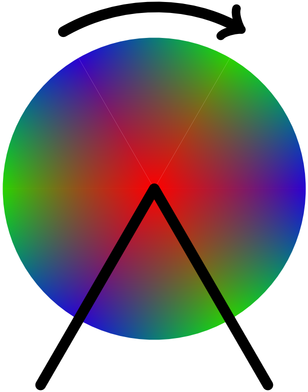

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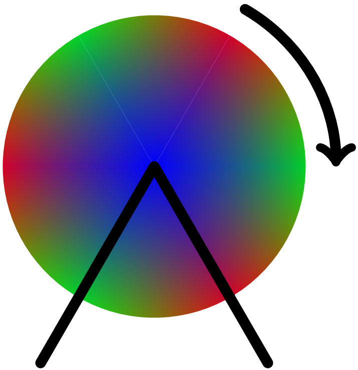

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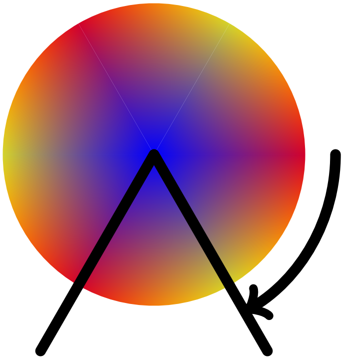

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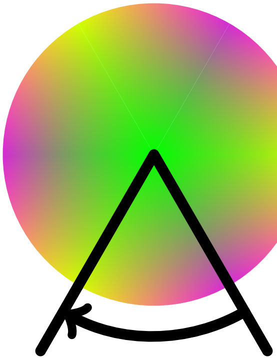

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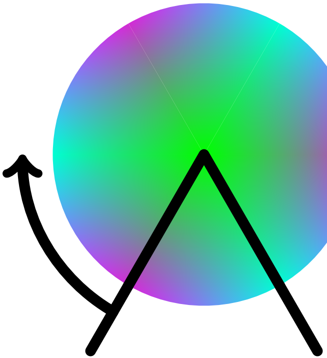

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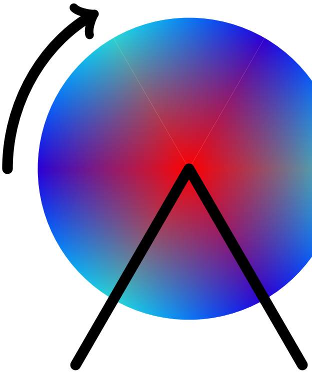

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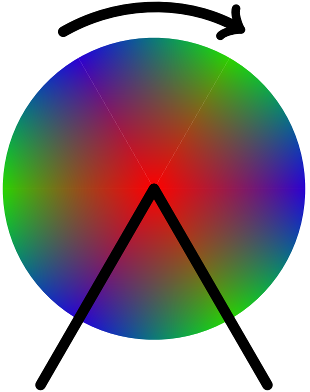

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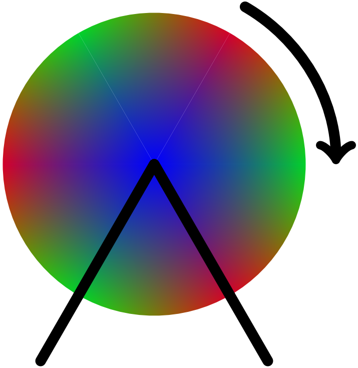

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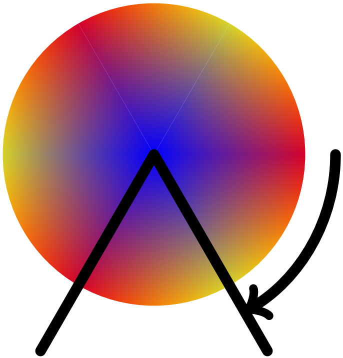

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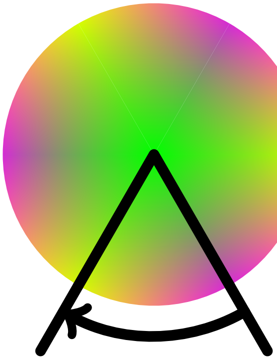

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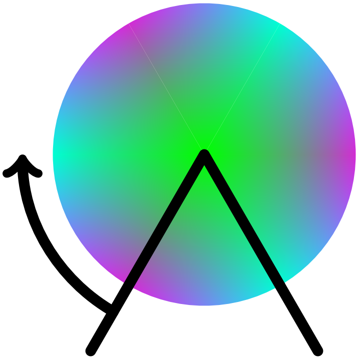

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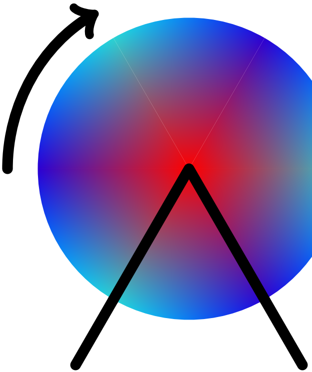

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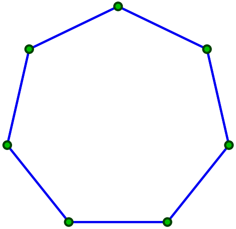

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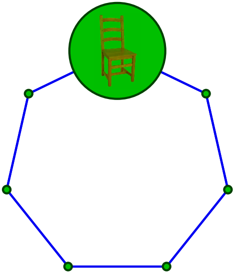

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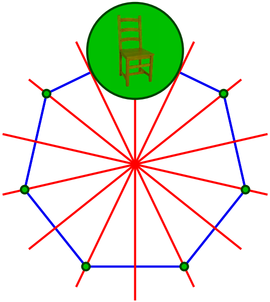

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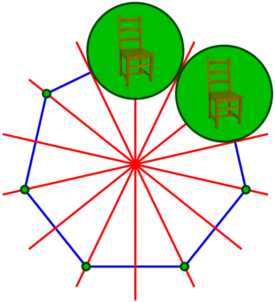

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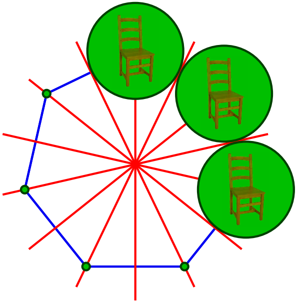

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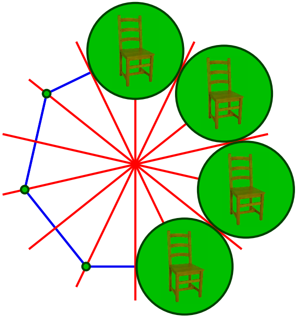

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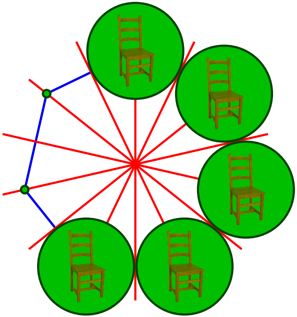

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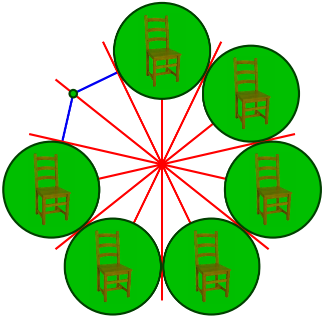

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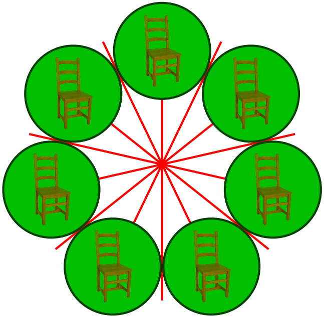

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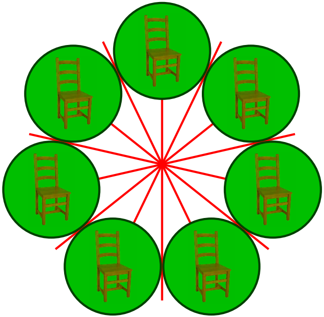

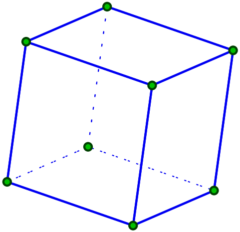

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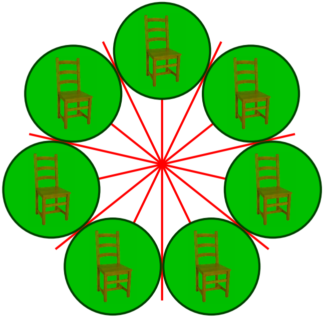

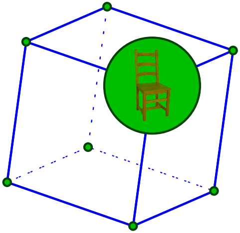

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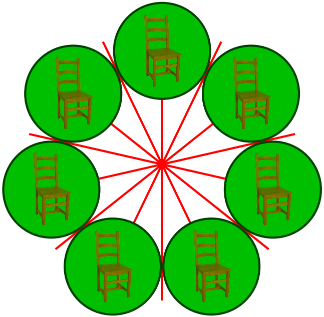

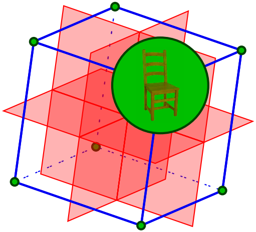

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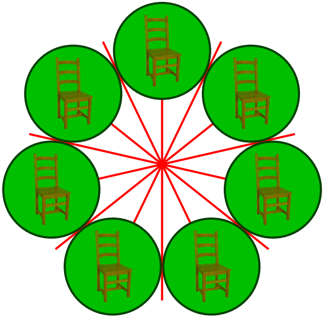

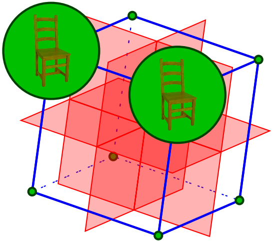

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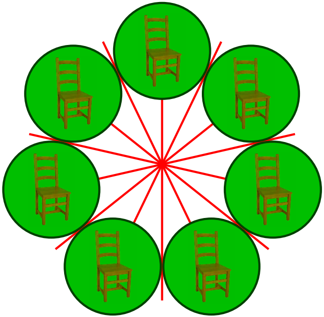

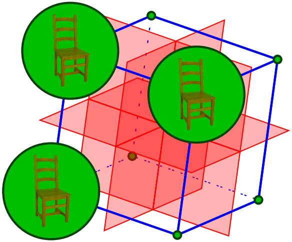

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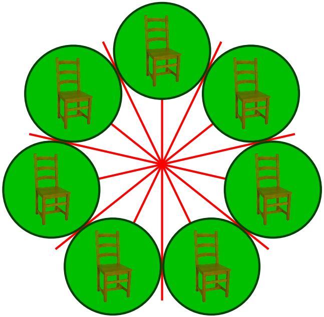

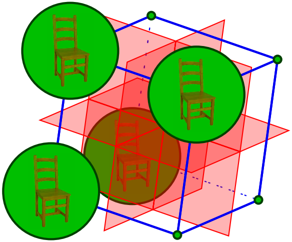

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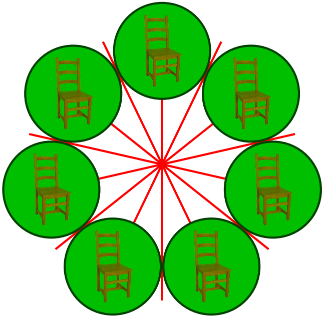

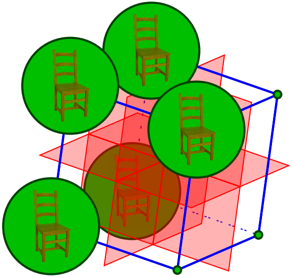

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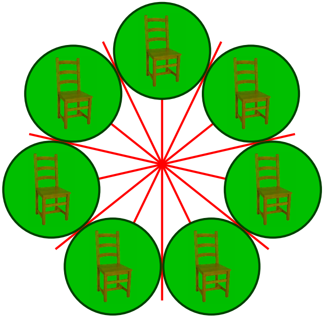

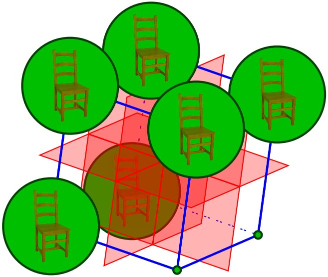

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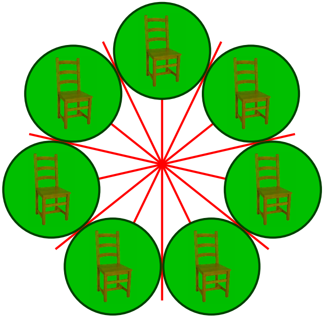

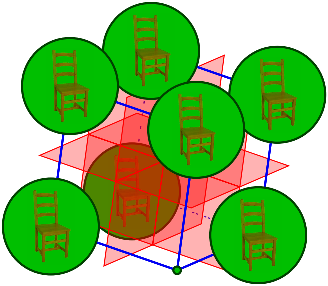

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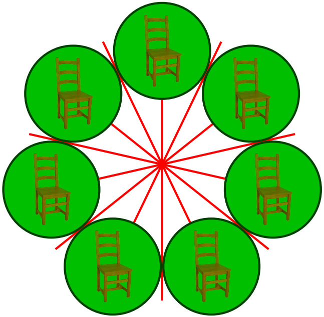

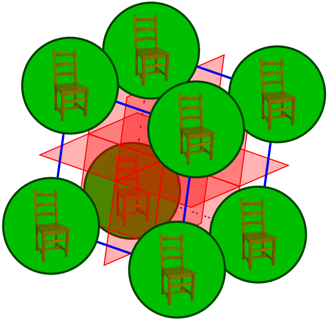

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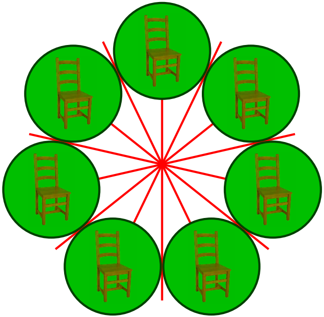
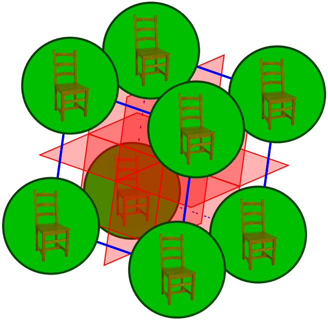
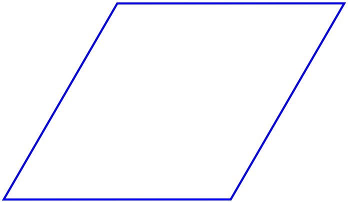




The number of mirror-images is finite

Kaleidoscope in higher dimensions



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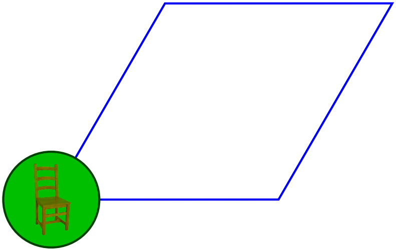

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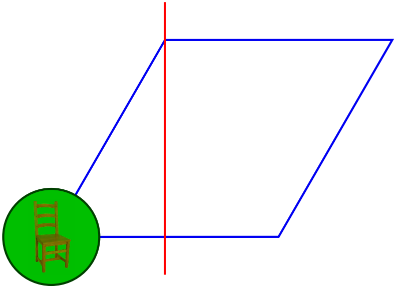

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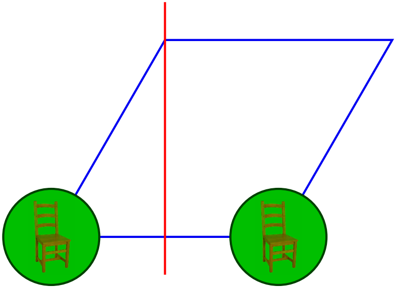

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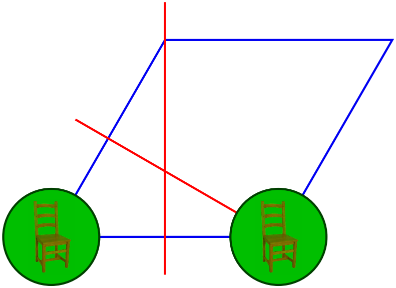

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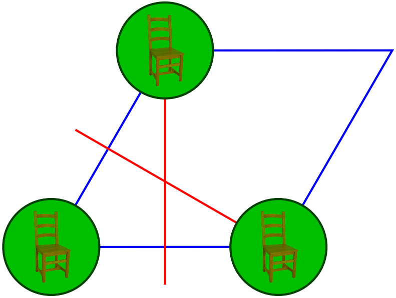

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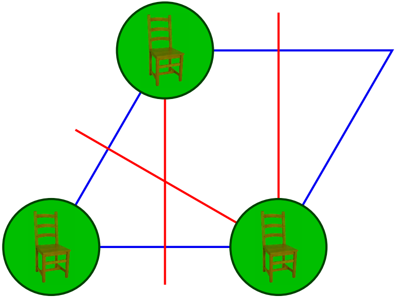

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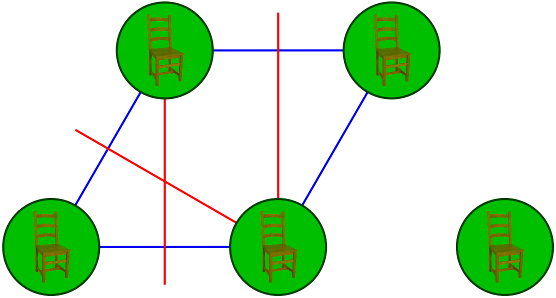

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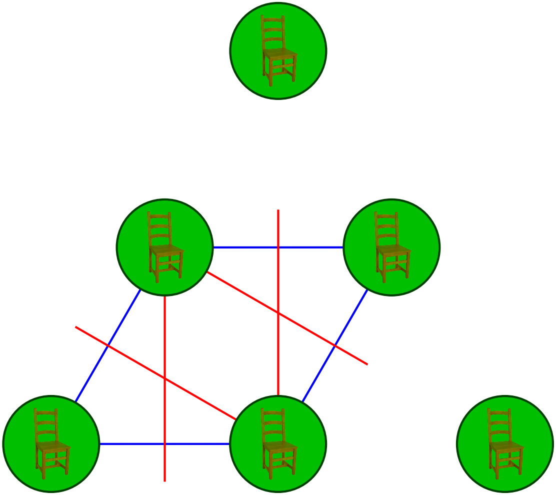

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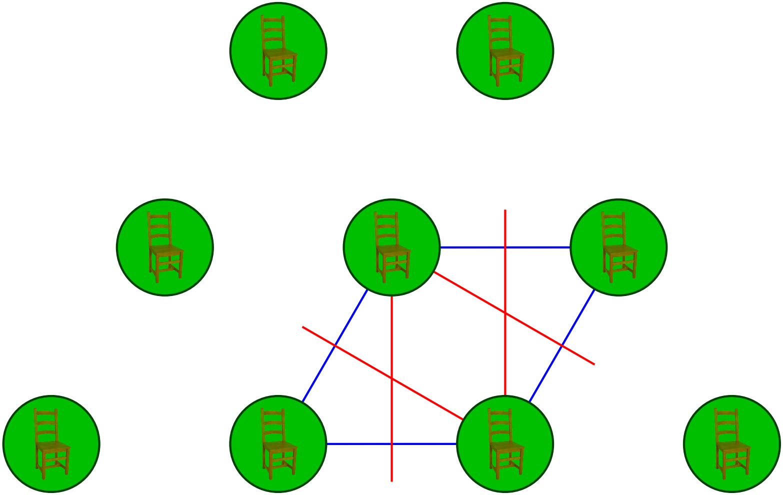

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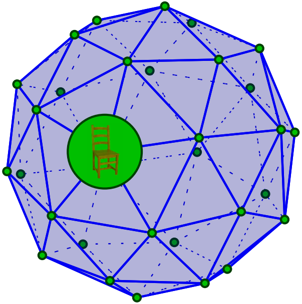

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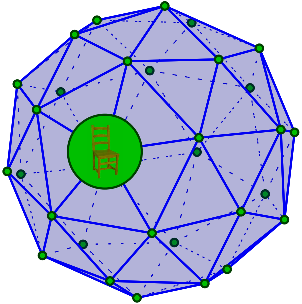

## Question

*When is the number of mirror-images finite?*

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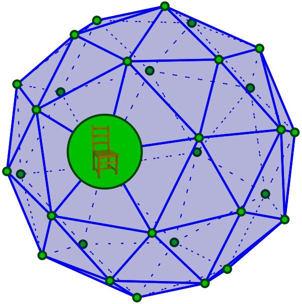

Question

*When is the number of mirror-images finite?*
*or equivalently: when is the exchange group $W(P)$ finite?*

# Bored already?

**Task**

*Obtain a non-regular pentagon with finite $W(P)$.*

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**Task (Bonus)**

*Obtain infinitely many non-regular pentagons with finite $W(P)$.*

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- Characterization of Gelfand–Serganova (Type $A$)
- Symmetric groups and matroids ($\rightsquigarrow$ Coxeter matroids)
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This is the object of Gelfand–Serganova's Theorem on Coxeter matroids.

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### Theorem (Coxeter (1934))

*Finite Euclidean reflection groups are exactly finite Coxeter groups.*

The classification: $A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, I_2(m)$.

Therefore, if $W(P)$ is finite, it is a subgroup of one of the above.

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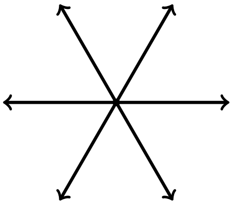

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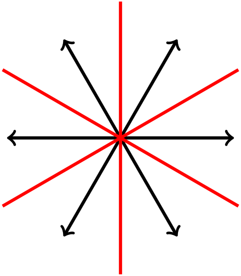

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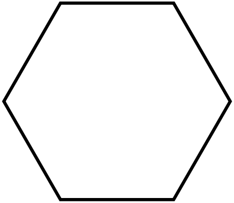

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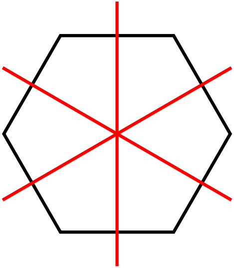

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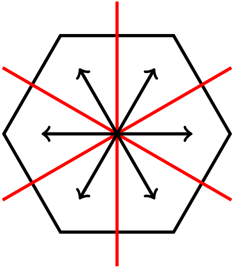

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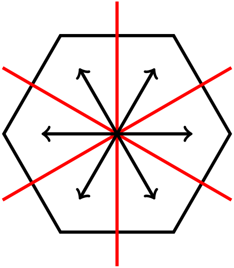

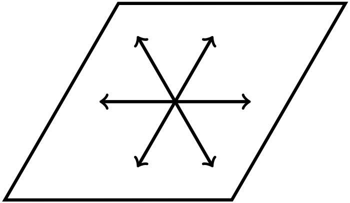

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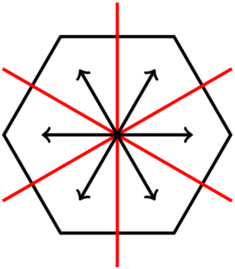

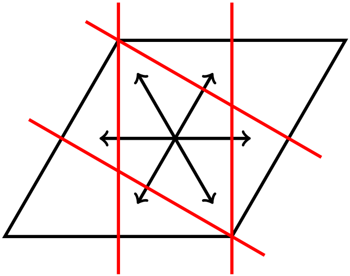

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# Matroids through Gale ordering

**Definition**

Let $\mathcal{P}_{n,k}$ be the collection of all $k$-element subsets in $[n]$ and

$$\begin{aligned} A &= \{i_1, \ldots, i_k\}, \quad i_1 < i_2 < \cdots < i_k, \\ B &= \{j_1, \ldots, j_k\}, \quad j_1 < j_2 < \cdots < j_k. \end{aligned}$$

Then $A \leq B \Longleftrightarrow i_1 \leq j_1, \ldots, i_k \leq j_k$.

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**Example**

Let $n = 5$ and $k = 3$, then $\{1, 3, 4\} \leq \{1, 3, 5\}$ and $\{1, 3, 5\} \not\leq \{2, 3, 4\}$.

# Matroids through Gale ordering

### Definition (Gale ordering induced by $w$)

Let $\mathcal{P}_{n,k}$ be the collection of all $k$-element subsets in $[n]$, $w \in \mathbb{S}_n$ and

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**Example**

Let $n = 5$, $k = 3$ and $w = [1, 2, 3, 5, 4]$ then $\{1, 3, 4\} \not\leq^w \{1, 3, 5\}$ and $\{1, 3, 5\} \leq^w \{2, 3, 4\}$.

# Matroids and Flag matroids – A primer

### Maximality Property

*For every $w \in \mathbb{S}_n$, the collection $\mathcal{B} \subseteq \mathcal{P}_{n,k}$ contains a unique member $A \in \mathcal{B}$ maximal in $\mathcal{B}$ with respect to $\leq^w$.*

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### Definition (Gale (1968))

Let $\mathcal{B} \subseteq \mathcal{P}_{n,k}$. Then $\mathcal{B}$ is a matroid if and only if $\mathcal{B}$ satisfies the Maximality Property.

# Examples of matroids

Let $n = 4$.

Example

For $k = 1$ and $\mathcal{B}_1 = \{1, 3, 4\}$

Example

For $k = 2$ and $\mathcal{B}_2 = \{12, 14, 23, 24, 34\}$

Example

For $k = 3$ and $\mathcal{B}_3 = \{123, 124, 134\}$.
For $w = [1, 2, 3, 4]$ the maximal element is 134.
For $w = [1, 3, 2, 4]$ the maximal element is 124.

# Flag matroids

A flag $F$ is a strictly increasing sequence

$$F_1 \subset F_2 \subset \cdots \subset F_m$$

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**Definition**

A collection $\mathcal{F}$ of flags is a flag matroid if and only if $\mathcal{F}$ satisfies the Maximality property:

*For every $w \in \mathbb{S}_n$ the collection contains a unique element in $\mathcal{F}$ with respect to the ordering $\leq^w$.*

# An example of flag matroid

Take again $n = 4$ and $\mathcal{B}_1 = \{1, 3, 4\}$, $\mathcal{B}_2 = \{12, 14, 23, 24, 34\}$, and $\mathcal{B}_3 = \{123, 124, 134\}$.

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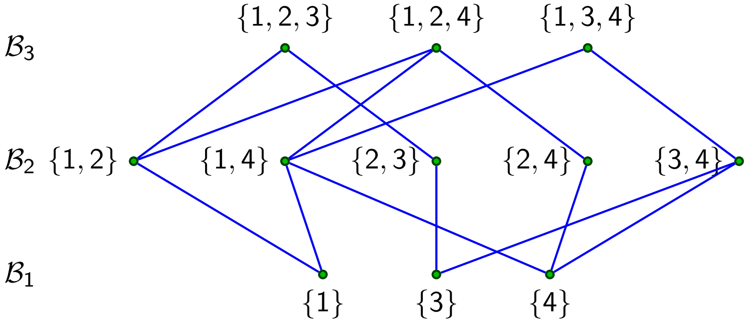

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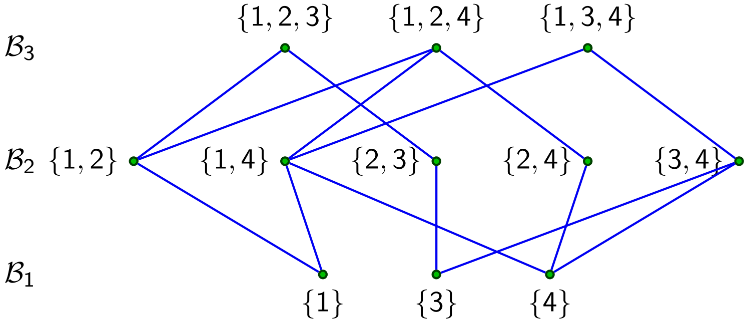


$\mathcal{F} = \{123, 124, 142, 143, 321, 341, 412, 413, 421, 431\}$

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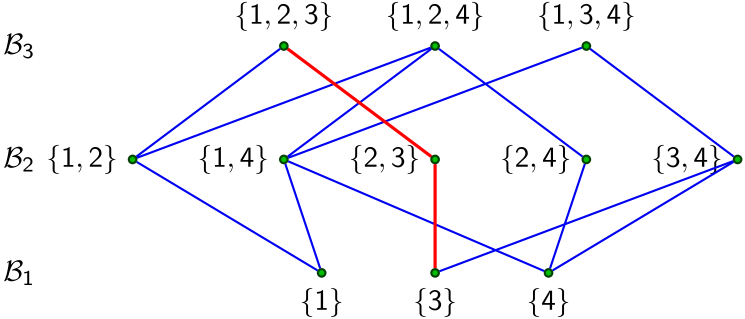


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Finally, for any set of flags $\mathcal{F}$, let

$$\Delta_{\mathcal{F}} = \mathrm{conv}\{\delta_F | F \in \mathcal{F}\}.$$

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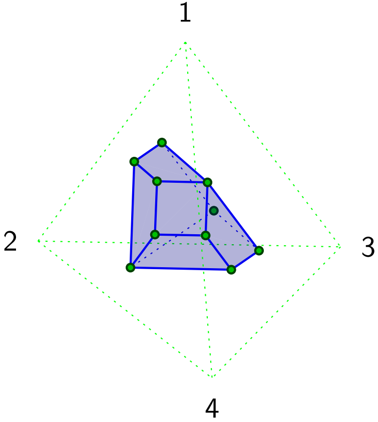


$\Delta_{\mathcal{F}} = \mathrm{conv}\{(3,2,1,0),(3,2,0,1),(3,1,0,2),(3,0,1,2),(1,2,3,0),$
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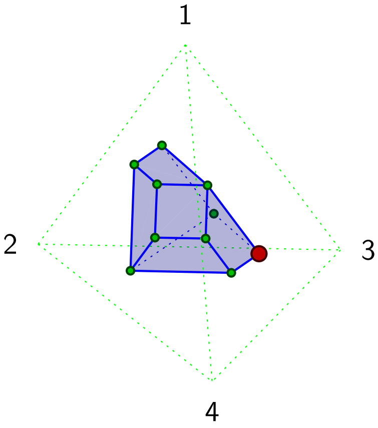


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# Gelfand–Serganova's Theorem (Type $A$)

**Theorem (Gelfand–Serganova, (1987))**

*Let $\mathcal{F}$ be a set of flags with cardinality $k_1, \ldots, k_m$ on $[n]$. The following conditions are equivalent:*

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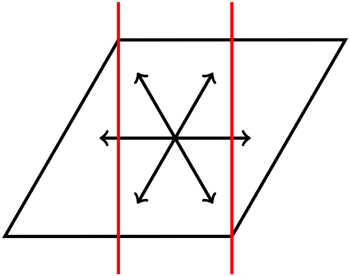
**Corollary**

*Let $P$ be a polytope. The following conditions are equivalent:*

1) $W(P) \leq \mathbb{S}_n$.
2) *$P$ is a $\Phi_{A_{n-1}}$-polytope with vertices equidistant to a point $p$.*

# Counter-example explained



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iii) The group $W(\Delta_{\mathcal{F}})$ is a reflection subgroup of $\mathbb{S}_n$

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ii) Prove the Increasing Exchange Property

iii) Prove that IEP $\implies$ Maximality Property

# Symmetric groups $\longleftrightarrow$ Matroids

Consider $\mathbb{S}_n$ and the subgroup $P_k$ generated by

$$(1\ 2), \ldots, (k-1\ \ k), \cancel{(k\ \ k+1)}, (k+1\ \ k+2), \ldots, (n-1\ \ n).$$

**Fact**

*$k$-subsets of $n$ $\longleftrightarrow$ cosets $\mathbb{S}_n/P_k$.*

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**Fact**

*$k$-subsets of $n$ $\longleftrightarrow$ cosets $\mathbb{S}_n/P_k$.*

**Example**

Let $n = 4$ and $k = 2$.

$$\begin{array}{ll}
\{1,2\}\longleftrightarrow[\{1{,}2\},\{3{,}4\}] & \{2,3\}\longleftrightarrow[\{2{,}3\},\{1{,}4\}] \\
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\end{array}$$

# Gale order $\longleftrightarrow$ Bruhat order

**Definition (Bruhat order)**

Let $u, v \in \mathbb{S}_n$. Then $u \leq_B v \iff u$ is a subword of $v$.

**Example**

$tut$ and $stusts$.

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### Theorem

*Let $\mathcal{F}_n^{k_1 k_2 \cdots k_m}$ be the set of flags of cardinality $k_1, k_2, \ldots, k_m$. Then*

$$(\mathcal{P}_{n,k}, \leq^w) \cong (\mathbb{S}_n / P_k, \leq_B^w)$$

*and*

$$(\mathcal{F}_n^{k_1 k_2 \cdots k_m}, \leq^w) \cong (\mathbb{S}_n / P_{k_1, k_2, \ldots, k_m}, \leq_B^w)$$

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**Definition**

A collection $\mathcal{M} \subseteq W/P$ is a Coxeter matroid for $W$ and $P$ if and only if it satisfies the Maximality Property:

*For any $w \in W$ the collection $\mathcal{M}$ contains a unique element in $\mathcal{M}$ with respect to the ordering $\leq_B^w$.*

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**Theorem (Gelfand–Serganova, (1987))**

*Let $\mathcal{M} \subseteq W/P$. The following conditions are equivalent:*

1) *$\mathcal{M}$ is a Coxeter matroid.*
2) $W(\Delta_{\mathcal{M}}) \leq W$.

# Solution to the Problem

Let $P$ be a $d$-dimensional polytope. The following conditions are equivalent

- $W(P)$ is finite.

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- The vertices of $P$ correspond to a collection of cosets forming a Coxeter matroid.

# Intermede – Solutions to the puzzles

**Task**

*Obtain a non-regular pentagon with finite $W(P)$.*

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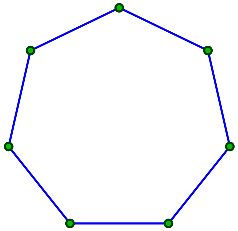

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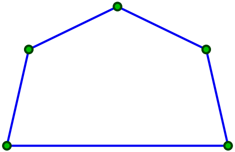

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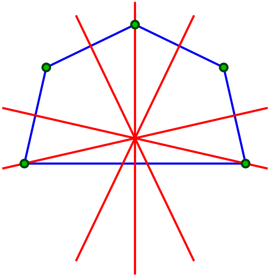

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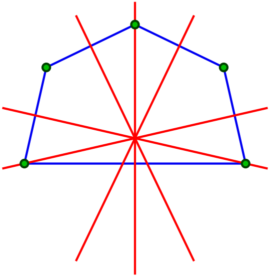
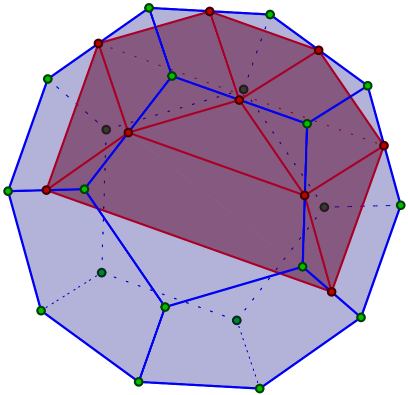


Task (Bonus)

*Obtain infinitely many non-regular pentagons with finite $W(P)$.*

# Merci! Thank you! Grazie! Danke! Gracias!



A $H_3$ matroid polytope

# Increasing Exchange Property

**Increasing Property**

*If $F_1$, $F_2$ are two different flags from $\mathcal{F}$ and $w \in \mathbb{S}_n$, then there is a transposition $t \in \mathbb{S}_n$ such that for one of the flags $F_1$, $F_2$, say $F_i$, $F_i <^w tF_i$ and $tF_i$ also belongs to $\mathcal{F}$.*