What are... Catalan numbers ?

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n	3	4	5	6	7	$\cdots$
Δ	1	2	5	14	42	$\cdots$

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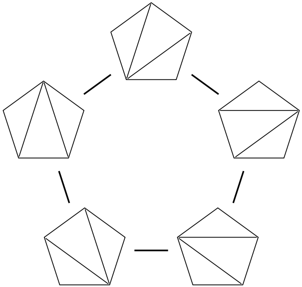

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Finally, Euler gave the following formula :

$$\frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdots (4n-10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot (n-1)}$$

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This number can be rewritten as

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Segner's recurrence formula

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Then, Euler essentially solved the recurrence though without giving a complete proof.

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These numbers are now known as Fuss-Catalan numbers.

Further developments

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Finally, Eugen Netto seems to have coined the name Catalan numbers in his book *Lehrbuch der Combinatorik* (1900).

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- Standard Young tableaux of shape $(n, n-1)$;
- Linear expansions of the poset $2 \times n$;

A simple geometric proof

Bijective proof using triangulations

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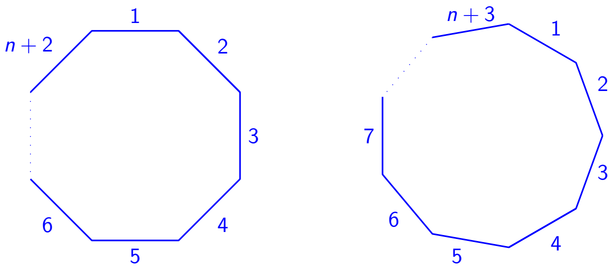

$(n+2)$-gon

$(n+3)$-gon

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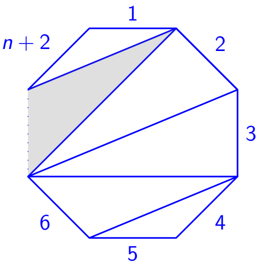

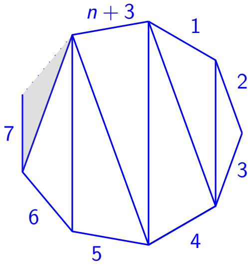

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C_n objects

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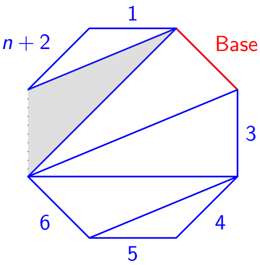

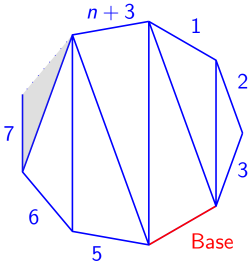

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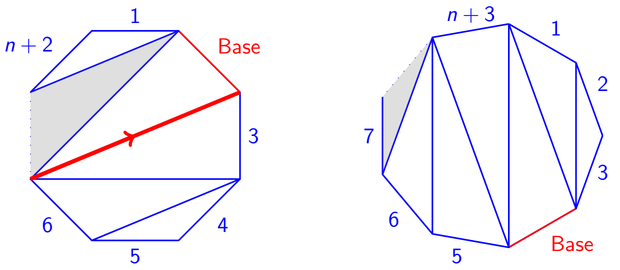

$(n+2)$-gon $(n+3)$-gon

$(4n+2)C_n$ objects C_{n+1} objects

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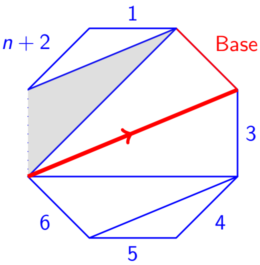

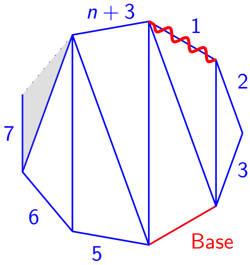

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$(n+2)C_{n+1}$ objects

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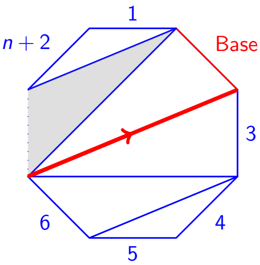

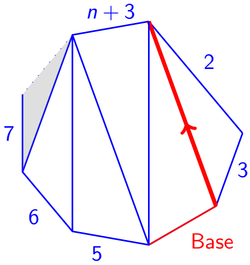

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A simple geometric proof - ... continued

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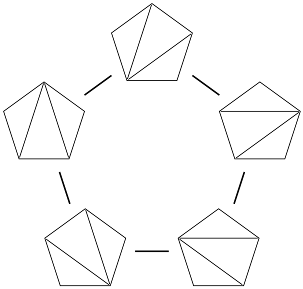
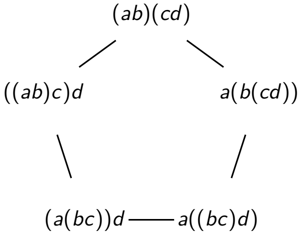
with $C_1 = 1$, we get the binomial formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

CQFD

A more *complicated* example





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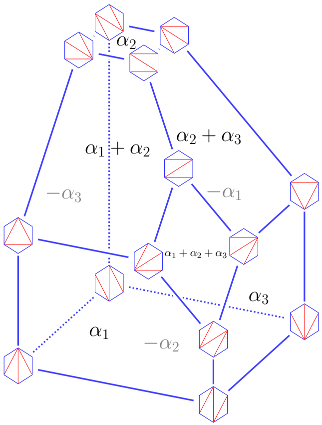

Reference

Richard Stanley, *Enumerative Combinatorics*, I-II, Cambridge Studies in Advanced Mathematics (1986).